What are *chimeras* in dynamical systems?

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Chimera
in Greek mythology

1. (2002) Yoshiki Kuramoto and Dorjsuren Battogtokh: Coherent–Incoherent dynamical regime [1]
2. (2004) Daniel Abrams and Steven Strogatz: name "chimera state" (chimera) [2]
3. (2012) Firs experimental observation of the chimera states

Chimera states appear in Mechanics, Chemistry, Biology, Neuroscience, Electronics, Optics, Sociology, electrochemistry and study of the financial market behaviour [3]

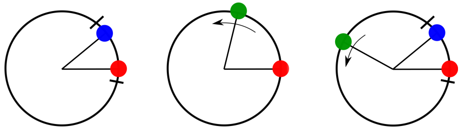
[1] Y. Kuramoto & D. Battogtokh, Coexistence of coherence and incoherence in nonlocally coupled phase oscillators, *Nonlinear Phenomena in Complex Systems*, **5(4)**, 380–385, (2002).
[2] D. Abrams & S. Strogatz, Chimera States for Coupled Oscillators, *Physical Review Lettters*, **93**, 174102, (2004).
[3] S. Zhao, Q. Xie, Q. Lu, X. Jiang, & W. Chen, Coherence and incoherence collective behavior in financial market, *EPL (Europhysics Letters)*, **112(2)**, 28002, (2015).

Kuramoto Model of coupled phase oscillators

$$\frac{d\theta_i}{dt} = \omega_i + \frac{1}{N}\sum_{j=1}^{N} K_{ij}\, g(\theta_i - \theta_j), \quad i = 1, \ldots, N, \tag{1}$$

$(\theta_1, \ldots, \theta_N) \in \mathbb{T}^N = [0, 2\pi)^N$ are phase variables
ω_i is natural frequency of ith oscillator
K_{ij} is the strength of coupling
$g(\phi)$ is smooth 2π–periodic coupling function

The phase oscillators are *identical* if $\omega_i = \omega$



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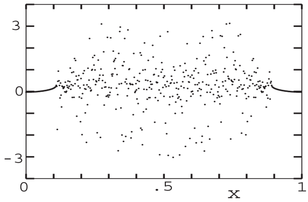
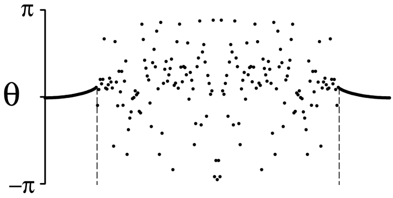
Approximate definition of the chimera state
Regime that for certain choices of parameters and initial conditions, the array would split into two domains: one composed of coherent, phase-locked oscillators, coexisting with another composed of incoherent, drifting oscillators.





Instantaneous spatial distribution of the phases (snapshot).
One point represents one oscillator $\theta_i \in \mathbb{T}^1$ that runs along its own vertical track. (Left) Kuramoto–Battogtokh, (Right) Abrams–Strogatz.

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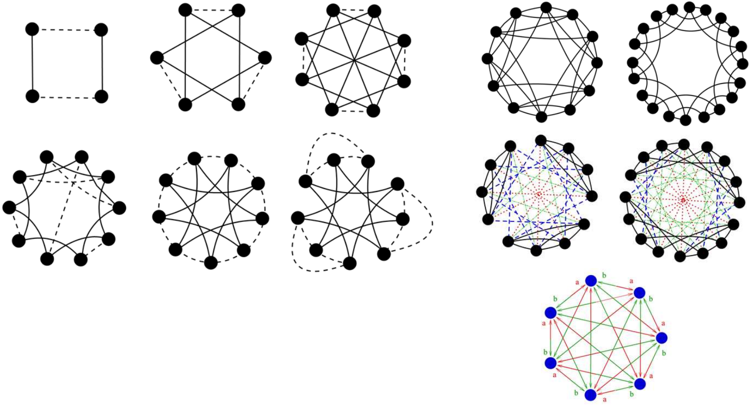
We consider *Hansel–Mato–Monier* coupling with parameters α and r:

$$g(\phi) = -\sin(\phi - \alpha) + r\sin(2\phi) \tag{2}$$

which is reduced to *Kuramoto–Sakaguchi* coupling when parameter $r = 0$

Indistinguishable phase oscillators

Definition 1
The oscillators are *indistinguishable* if the oscillators are identical and interchangeable in the sense that they have the same number and strength of inputs.

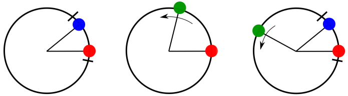


Definition 2
Oscillators i and j on the trajectory of the system (1) are *frequency synchronized* if

$$\Omega_{ij} := \lim_{T\to\infty} \frac{1}{T}[\theta_i(T) - \theta_j(T)] = 0$$

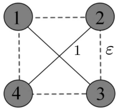
where we chose continuous representation of $\theta_i(t)$, $\theta_j(t)$.

Definition 3 (Weak Chimera [1])
$A \subset \mathbb{T}^N$ is a *weak chimera state* for a coupled phase oscillator system, if it is connected chain–recurrent flow–invariant set such that on each trajectory within A there are i, j and k such that $\Omega_{ij} = 0$ and $\Omega_{ik} \neq 0$.



[1] P. Ashwin & O. Burylko,
Weak chimeras in minimal networks of coupled phase oscillators,
Chaos, **25(1)**, 013106, (2015).

Four oscillators: Stable weak chimera with in–phase and anti–phase groups



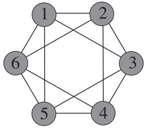
$$\frac{d\theta_1}{dt} = \omega + g(\theta_1 - \theta_3) + g(0) + \varepsilon g(\theta_1 - \theta_2),$$
$$\frac{d\theta_2}{dt} = \omega + g(\theta_2 - \theta_4) + g(0) + \varepsilon g(\theta_2 - \theta_1),$$
$$\frac{d\theta_3}{dt} = \omega + g(\theta_3 - \theta_1) + g(0) + \varepsilon g(\theta_3 - \theta_4),$$
$$\frac{d\theta_4}{dt} = \omega + g(\theta_4 - \theta_2) + g(0) + \varepsilon g(\theta_4 - \theta_3), \qquad (3)$$

$$g(\phi) = -\sin(\phi - \alpha) + r\sin(2\phi) \qquad (2)$$

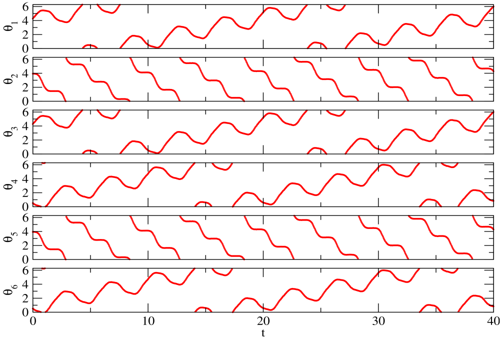
Theorem

There is an open set of (r, α) such that four–oscillator system (3), (2) has an attracting weak chimera state for $\varepsilon = 0$ that persists for all ε with $|\varepsilon|$ sufficiently small.

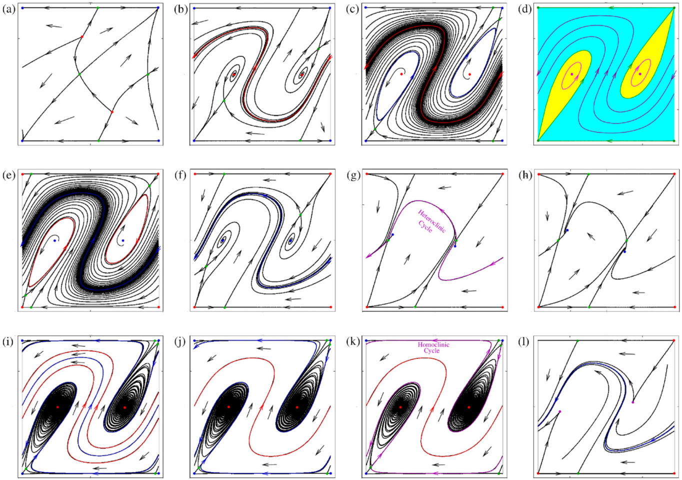
Week chimera states in *ring* network of six phase oscillators



$$\frac{d\theta_i}{dt} = \omega + \frac{1}{N} \sum_{|i-j|=1,2} g(\theta_i - \theta_j), \quad i = 1, \dots, 6,$$

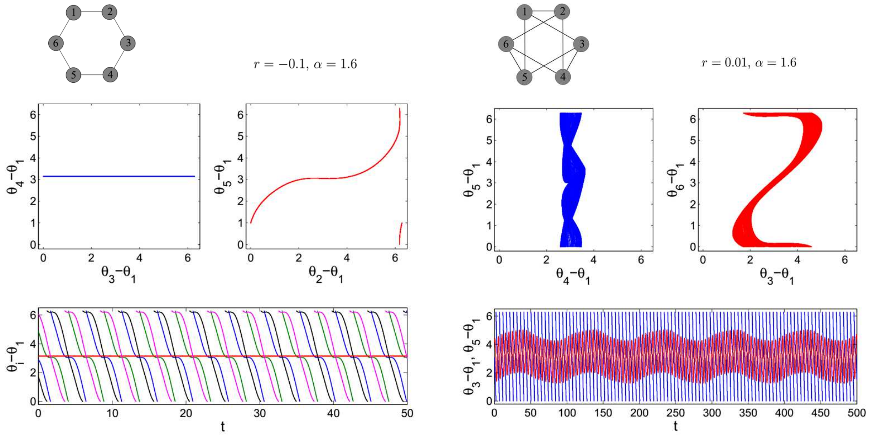


A stable weak chimera state in the ring of six phase oscillators showing for $\alpha = 1.56$, $r = -0.1$. The solution belong to invariant subspace $\mathcal{A}_1 \supset \mathcal{A}_7 : (\theta_1, \dots, \theta_6) = (a, b, a, a + \pi, b, a + \pi)$.



Pase portraits for the reduced system in variables $\xi = \theta_i - \theta_j \in [0, 2\pi)$, $\eta = \theta_i - \theta_k \in [0, 2\pi)$ for different values of parameter. The periodic, homoclinic and heteroclinic orbits that wind around ξ direction of the torus $\mathbb{T}^2$ are *weak chimera states*.

Weak chimera states in two networks of six oscillators



Projections and time–series of chimera solutions:
(left) periodic; (right) quasiperiodic

Summary

- *Coherent–incoherent regime* or *chimera state* was observed and proposed to consideration by Kuramoto and Battogtkh in 2002
- *Chimera state* is a combination synchronization and desynchronization in the network of *indistinguishable* elements
- There is a strong analytical definition of the *weak chimera state*
- There must be at least two synchronous (pase locked) and two desynchronous (phase unlocked) oscillators in the network for the chimeras existence
- The minimal system in the network of phase oscillators is *four* for the *weak chimera state*
- The minimal system in Kuramoto model with inertia is *three* for the *chimera*
- *Chimera state* can be: stable, unstable, neutrally stable, transitive, quasiperiodic, chaotic, heteroclinic
- The first physical experiments that proved the existence of the *Chimeras* in the nature were in 2012
- *Chimeras* can be observed for the networks of very different elements (pendulums, lasers, neurons, etc.)