Chaos, global indeterminacy and chaos control in the NK model

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Playground and main idea

- We consider the **New Keynesian (NK) optimal control model in continuous time**
- We show that an **active monetary policy**, combined with a **Ricardian passive fiscal policy** may induce **the onset of a chaotic attractor** in the region of the parameter space where **uniqueness** of the **equilibrium locally prevails**.
- We discuss **some implications**
- - **global indeterminacy**
- - **chaos control** (**policy devising)**

Consider the **optimization problem** faced by a **household-firm** i in **sticky-price NK model(money-in-the utility-function)** in **continuous time**

$$\underset{c_i, m_i, l_i}{Max} \int_0^\infty \left[u(c_i, m_i) - f(l_i) - \frac{\eta}{2}(\pi_i - \pi^*)^2 \right] e^{-\rho t} dt \quad (\mathcal{P})$$

subject to

$$\begin{aligned} \dot{a}_i &= [R - \pi_i] a_i - Rm(c_i, R) + \frac{p_i}{p} y(l_i) - c_i - \tau \\ \dot{p}_i &= \pi_i p_i \\ a_i(0) &= a_{i0} \\ p_i(0) &= p_{i0} \end{aligned}$$

Notation

- $u(c_i, m_i)$: **utility** derived from **consumption,** c_i, and from real **money balances**, m_i, $u \in C^2$ with

$$u_c(c_i, m_i) > 0; \quad u_{cc}(c_i, m_i) < 0; \quad u_m(c_i, m_i) > 0; \quad u_{mm}(c_i, m_i) < 0$$

- $f(l_i)$: **disutility of labor** (l_i) with $f(l_i) \in C^2$, with

$$f_l > 0; \quad f_{ll} < 0$$

- p : is the aggregate **price level**
- $\frac{\eta}{2}(\pi_i - \pi^*)^2$: **disutility of deviations of price** change $(\pi_i = \frac{\dot{p}_i}{p_i})$ from **intended rate** π^*
- $y(l_i)$ is an **endowment of perishable goods**, which is produced according to a **production function** using only **labour. sales of good** i are demand determined, so that

$$y(l_i) = \left(\frac{p_i}{p}\right)^{-\phi} y^d$$

where $\phi > 1$ is the **elasticity of substitution** across varieties.
- a : real **government liabilities**

Notation

- τ : lump-sum **taxes**
- R : **nominal interest rate**
- η : **measures** the degree to which **household-firms dislike to deviate** in their price-setting behavior from the the **intended rate of inflation**, π^*
- ρ : **discount factor**

The implied dynamics

Solutions of problem $\mathcal{P}$ satisfy the following **first order three dimensional system** of odes (cf. Benhabib et al. 2001)

$$\begin{aligned}
\dot{\mu}_1 &= [\rho - R + \pi]\,\mu_1 \qquad (\mathcal{M})\\
\eta\dot{\pi} &= \rho(\pi - \pi^*)\eta - y(l(\mu_1,\pi))\mu_1\left[(1-\phi) + \phi\frac{f'(l(\mu_1,\pi))l'(\mu_1,\pi)}{\mu_1 y'(l(\mu_1,\pi))}\right]\\
\dot{a} &= [R - \pi]\,a - Rm(y(l(\mu_1,\pi)),R) + \tau
\end{aligned}$$

where μ_1 is **shadow price of real balances (co-state variable)**,

- The **first equation** denotes the time evolution of the **Lagrange multiplier associated with the instant budget constraint** (or **shadow price** of the **real value of aggregate per capita government liabilities - real balances** and **bonds**).
- The **second equation** is referred to as the **New Keynesian Phillips Curve**.
- The **third equation** reproposes the **instant budget constraint**.
- **Solutions of system** $\mathcal{M}$ are admissible if the **no-Ponzi TVC condition**

$$0 = \lim_{t \to \infty} e^{-\int_0^t [R(s) - \pi(s)] ds} a(t)$$

is **satisfied**.

Behavior of public authority

Assumption 1 (Zero lower bound on nominal rates and Taylor principle). **Monetary authorities** *set the* **nominal interest rate** *as an increasing function of the inflation rate, implying that*

$$R = R(\pi) > 0; \qquad R'(\pi) > 0$$

It is assumed that there exists an inflation rate π^* at which the steady-state Fisher equation is satisfied. Namely

$$R(\pi^*) = \bar{R}$$

Assumption 2 Money and consumption *are Edgeworth substitutes in the* **utility function**. *Therefore*

$$u^*_{cm} < 0$$

Some useful definitions

Definition

(cf. Benhabib *et al.*, 2001)
i)Let $R'(\pi) > 1$. (**active monetary policy**)Then the **policy maker** reacts **more than proportionally** to an **increase in the inflation rate** .
ii)Let $R'(\pi) < 1$, (**passive monetary policy**).Then the **policy maker** reacts **less than proportionally** to an **increase** in the **inflation rate**

Definition

(cf. Leeper, 1991; Kumhof *et al.*, 2010)
i)Let $\tau'(a) > R(\pi) - \pi$. (**passive fiscal policy).**Then,the **dynamic path** of **total government liabilities** is **stable**
ii)Let $\tau'(a) < R(\pi) - \pi$ (**active fiscal policy).**Then the dynamic path of **total government** liabilities is **unstable**.

Local stability properties

Let now **J** denote the **Jacobian matrix** of system $(\mathcal{M})$, evaluated at the steady state $P^* \equiv P^*(\mu_1^*, \pi^*, a^*)$.

Lemma

(**Local stability properties** *of* P^*). *We assume:* $u_{cm}^* < 0$ *and* **active monetary policy** *then* **two stability cases** *can occur according to the magnitude of* $|u_{cm}^*|$.

a)Case $|u_{cm}^*| < |\hat{u}_{cm}^*|$, *where* $\hat{u}_{cm}^*$ *is a* **critical threshold**. *Then:*

a_i*)if* **fiscal policy is also active**, P^* **is a repellor** *and there are* **no equilibrium paths besides the steady-state** *itself*;

a_{ii}*)if* **fiscal policy is passive**, P^* *is a* **saddle of index** 2 *and the* **equilibrium is locally unique**.

Consider now

b)Case $|u_{cm}^*| > |\hat{u}_{cm}^*|$;

b_i*)if* **fiscal policy is passive**, P^* *is an* **attractor**

b_{ii}*)if* **fiscal policy is active** *there is a* **continuum of equilibria converging to the steady-state** (*local indeterminacy*).

Shilnikov chaos

Lemma

.(*Chen and Zhou*, 2011). *Consider* $\frac{dY}{dt} = f(Y, \alpha)$, $(*)$ $Y \in \mathbb{R}^3$, $\alpha \in \mathbb{R}^1$ *with* f *smooth.*

The eigenvalues of the Jacobian $A = Df$ *are:* γ *and* $\chi \pm \xi i$,$\gamma\chi < 0$ *in the equilibrium point* Y_0.

$(H.1)$ *the* **saddle quantity** $SQ \equiv |\gamma| - |\chi| > 0$;

$(H.2)$ *there exists a* **homoclinic orbit** Γ_0 *based at* Y_0.

Then:(1) the Shilnikov map, defined in the neighborhood of the homoclinic orbit Γ_0 *possesses an infinite number of Smale horseshoes in its discrete dynamics;*

(2) for any sufficiently small C^1*-perturbation* g *of* f, *the perturbed system has at least a finite number of Smale horseshoes in the discrete dynamics of the Shilnikov map, very close to* Γ_0;

(3) both the original and the perturbed system exhibit horseshoe chaos.

Standard assumptions

$$u(c,m) = \frac{\left[\kappa c^{1-\beta} + (1-\kappa) m^{1-\beta}\right]^{\frac{1-\Phi}{1-\beta}}}{1-\Phi}, \; f(l) = \frac{l^{1+\psi}}{1+\psi}, \quad y(l) = Al$$

$$R(\pi) = \bar{R} e^{(C/\bar{R})(\pi-\pi^*)}, \;\; R(\pi^*) = \bar{R}, \qquad R'(\pi^*) = C \;, \tau(a) = \alpha a - Rm$$

where:

- k : **share parameter**, $0 < \kappa < 1$
- Φ : **inverse of the intertemporal elasticity of substitution**, $\Phi > 0$
- ψ : **measures** the preference **weight of leisure** in **utility** $\psi > 0$
- β : intra-temporal **elasticity of substitution** between the **two arguments**, **c** and **m**
- C : is a **constant**
- α : is the **marginal fiscal rate**.

- The application of Lemma 1 to system $\mathcal{M}$ requires that several conditions be fulfilled:
- *i*) system $\mathcal{M}$ possesses a **hyperbolic saddle-focus** equilibrium point P^*;
- *ii*) in the case of the saddle-focus equilibrium P^*, SQ **be positive**;
- *iii*) in the case of the saddle-focus equilibrium P^*, with $SQ > 0$, there **exists a homoclinic orbit**, joining the saddle-focus $P^*$to itself.

Denote the set of the "deep" parameters of the economy as $\mathcal{D} \equiv (\beta, \eta, \kappa, \phi, \psi, \rho, \Phi)$ and assume

$$\bar{\mathcal{D}} \equiv (1.975, 350, 0.78966, 21, 1, 0.018, 2) \in \mathcal{D}$$

Set furthermore the pair $(\bar{R}, \pi^*) = (0.06, 0.042)$ (**inflation rate** observed **in the period from 1960 to 1998 by the US economy)**.
Since τ **simplifies out in the calculations**, the **characteristic equation is a function of the remaining policy parameters** C and the **marginal tax rate** τ'.

Therefore

$$\lambda_1 = 0.018 - \tau'$$
$$\lambda_{2,3} = 0.009 - 0.00058C \pm 0.00058\sqrt{(C - 1.00046)(C - 4.6543 \times 10^5)}$$

and

$$SQ \equiv \tau' - 0.027 + 0.00058C$$

If we set

$$\tau' > 0.027 - 0.00058C = \tilde{\tau}'$$

the **saddle quantity** SQ **is positive** .

An example

Proving precondition **H.2** in **Theorem 1** is not easy. Steps:

- **translation to the origin**;
- computation **normal form** using eigenbasis
- computation **split function**

Split function Σ

$$\Sigma = \Xi + \frac{F_{3f}\Xi^2}{\gamma} + (2\chi - \gamma)\frac{F_{3a}\Psi\Omega + F_{3d}\Psi^2 + F_{3e}\Omega^2}{(2\chi-\gamma)^2 + 4\xi^2} = 0$$

where $(\Xi, \Psi, \Omega) \in (0,1)^3$ are **free constant** and

$$\gamma = \lambda_1, \chi = \operatorname{Re}\lambda_{2,3} \text{ and } \xi = \operatorname{Im}\lambda_{2,3}$$

and the $F_{i,j}$ **coefficients**, with $i = 1,2,3$ and $j = a, b...f$, are **intricate combinations of the original parameters** of the model. Given parameters, conditions for the **existence of the homoclinic loop doubly** asymptotic to the **saddle-focus equilibrium point** rely on the existence of a **triplet** $(\Xi, \Psi, \Omega) \in (0,1)^3$ satisfying $\Sigma = 0$ (**admissible solution**).

Let $\mathcal{D} = \bar{\mathcal{D}}$ and $(\bar{R}, \pi^*) = (0.06, 0.042)$ as in Example 1. Set $C = 1.5$.
Let $\tau' = \tilde{\tau}'$

$$\tilde{\tau}' = 0.027 - 0.00058C = 0.02613$$

the **critical value of the marginal tax rate** τ'at which the **saddle quantity** SQ is positive.
Then,
i) if $\tau' > \tilde{\tau}'$, P^* is a **saddle-focus** with **positive saddle quantity** $SQ > 0$
ii)We use the **marginal tax rate** τ' **as the bifurcation parameter**
iii)We **iteratively** increase **marginal tax rate** τ' above 0.02613 with a grid of 0.01, till a solution for $\Sigma = 0$ with $(\Xi, \Psi, \Omega) \in (0, 1)^3$ emerges the procedure reveals that $\exists\ I_{\tau'} \cong (0.02613, 0.23543)$ such that $\forall \tau' \in I_{\tau'}$ a **family of homoclinic loops doubly asymptotic to the saddle-focus equilibrium point exists.**

The following statement is therefore implied.

Lemma

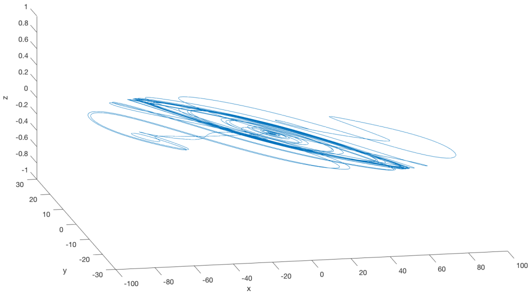
. There **exist regions** *in the* **parameter space** *such that pre-conditions H.*1 *and H.*2 *in Lemma* 2 *are* **simultaneously satisfied**.

Theorem

(*Existence of a Shilnikov chaotic attractor*). *Assume parametric conditions in Lemma* 2 *and* 3 *are satisfied. Then, given a triplet of initial conditions* $(w_1(0), w_2(0), w_3(0))$ *sufficiently close to the origin, system* $(*)$ *admits perfect-foresight chaotic equilibrium solution. By* **topological equivalence**, *the result also applies to system* $\mathcal{M}$.

Then, by **Shilnikov Theorem**, the **dynamics exhibits horseshoes chaos**.

- The **attractor** is represented in Figure .



Chaos control and policy

The methods of controlling chaotic dynamics provide useful tools in this regard: under certain conditions, **undesired irregular or even cyclical behavior can be switched off.**

- Method: Ott, Grebogi and Yorke (1990) (**OGY algorithm**). Force a **chaotic trajectory** onto a **desired target**
- The **control parameter** has to be accessible to the **decision makers**. We have chosen the **nominal interest rate** $\bar{R}$
- It becomes quite natural to allow the **policy-maker** to announce a **tightening or a loosening of the nominal rate** (at the intended **steady-state**) in order **to achieve chaos control**.

Chaos control and policy

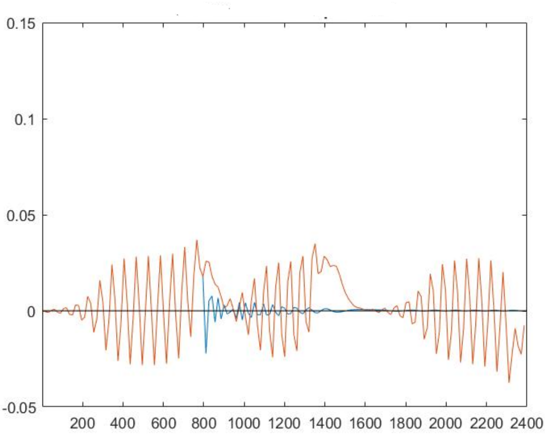
Lemma

System $\mathcal{M}$ satisfies the conditions for controllability.

Theorem

Consider the case the **policy-maker runs** *an* **active-passive monetary-fiscal regime** *and assume $\bar{R} \in \bar{R}^{+}_{\Theta}$. Then, by Theorem1,* **one eigenvalue is negative and two eigenvalues have positive real parts.** *Assume furthermore the* **economy evolves within a chaotic attractor**. *Consider now the* **policy-make**r *announces to commit to a* **higher steady state nominal interest rate belonging to the set** $(\bar{R})^{-}_{\Theta}$. *Then, the* **economy supersedes irregular and cyclical behavior and approaches the intended steady state.**

Chaos control and policy



- The NK model **exhibits global indeterminacy of the equilibrium**.
- In the NK model **relative frequency** at which an **orbit visits different regions** of the attractor **is largely heterogenous**.
- Then, across the volume of **all possible coordinates** contained in the **attractor**, the **economy lingers on particular regions with higher "density".**
- **Chaos Control**. In the NK model the **economy** can **supersedes irregular and cyclical behavior** and **approaches the intended steady state.**

Contacts

Thank you for your attention!

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