On Monotone Equilibria of an Asymmetric First-Price Auction with Affiliated Private Information

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Bayesian Games with Continuous Type Spaces

Behavioral strategies:
Lebrun (1998), He and Yannelis (2015, 2016),
Carbonell-Nicolau and McLean (2018)

Pure strategies: Monotone equilibria :
Athey (2001), Zandt and Vives (2007)
Reny and Zamir (2004), Reny (2011)

The Model

A Bayesian game $\Gamma = (B_i, T_i, f, v_i)_{i \in I}$:

(1) the set of players $I = \{1, \ldots, n\}$;

(2) bidder i's set of bids $B_i = [0, c]$, $c \geq 1$;

(3) bidder i's set of types $T_i = [0, 1]$;

(4) the joint continuous density of the bidders' types $f : T \to (0, +\infty)$

(5) player i's payoff function $v_i : B \times T \to \mathbb{R}$

Payoff functions

$$v_i(b;t) = \begin{cases} \frac{1}{m} u_i(b_i;t) \text{ if } m = \#\{j \in I : b_j = b_i = \max_{k \in I} b_k\} \geqslant 1, \\ 0 \text{ otherwise,} \end{cases}$$

where

$u_i : B_i \times T \to R$ possesses the following properties:

(i) u_i is jointly continuous in $(b_i;t)$, nonincreasing in b_i, and nondecreasing in t_j for each $j \in I$;

(ii) $u_i(0;t) \geq 0$ for all $t \in T$ and $u_i(c;t) < 0$ for all $t \in T$;

(iii) u_i has increasing differences in (b_i, t_i) (independent types or $n = 2$) or in (b_i, t) (many players with affiliated types)

Affiliated Types

Assumption B.1. The function $f : T \rightarrow (0, +\infty)$ satisfies the following log-supermodularity condition: $f(t \wedge t')f(t \vee t') \geq f(t)f(t')$ for all $t, t' \in T$,
where $\wedge$ and $\vee$ denote the componentwise minimum and maximum

Assumption B.2 (Independent types). $f : T \rightarrow (0, +\infty)$ satisfies the following condition:
$f(t) = f_1(t_1) \times \ldots \times f_n(t_n)$ for all $t \in T$.

Strategies and Payoffs

L_i the set of Lebesgue measurable functions from T_i into B_i

S_i the set of nondecreasing functions from from T_i to B_i.

$d_1(s_i, s_i') = \int_{T_i} \left| s_i(t_i) - s_i'(t_i) \right| dt_i$ for all s_i, $s_i' \in L_i$ (or S_i)

Trader i's interim payoff function

$$V_i(b_i, s_{-i}; t_i) = \int_{T_{-i}} v_i(b_i, s_{-i}(t_{-i}), t_i; t_{-i}) f_{-i}(t_{-i}) dt_{-i}.$$

Trader i's interim value function

$$\overline{V}_i(s_{-i}; t_i) = \sup_{b_i \in B_i} V_i(b_i, s_{-i}; t_i).$$

H-**Spaces**

Each S_i is considered as an H-space.
An H-space is a pair $(X, \{F_A\})$,
where X is a topological space;
$\{F_A\}$ is a family of nonempty contractible subsets of X,
indexed by the finite subsets of X,
such that $A \subset B$ implies $F_A \subset F_B$

A nonempty subset D of X is H-convex
if $F_A \subset D$ for each finite subset A of D.

H-spaces of Nondecreasing Strategies

For a finite family of strategies $A = \{s_{1i}, \ldots, s_{ki}\} \subset S_i$,
F_A as the minimal subset of S_i with the following properties:
(i) it contains A;
(ii) if $s_i, s_i' \in F_A$, then $s_i \vee s_i' \in F_A$;
(iii) for every s_i and s_i' are in F_A and every $\tau \in (0,1)$,
$s_i \mathcal{X}_{[0,1-\tau]} + (s_i \vee s_i')\mathcal{X}_{(1-\tau,1]}$ is in F_A,
where $\mathcal{X}_D$ denotes the indicator function of the set D
in (iii) piecewise combinations of the two functions

Interim ε-Equilibria

A strategy profile $s \in L$ is an interim ε-equilibrium ($\varepsilon > 0$) of Γ if, for each i and for almost all $t_i \in T_i$,
$V_i(s_i(t_i), s_{-i}; t_i) > \overline{V}_i(s_{-i}; t_i) - \varepsilon$
Bidder i's ex-ante payoff function $V_i^* : \Pi_{i=1}^n L_i \longrightarrow \mathbb{R}$ is defined by

$$V_i^*(s) = \int_{T_i} V_i(s_i(t_i), s_{-i}; t_i) f_i(t_i) dt_i.$$

Bidder i's ex-ante value function $\overline{V}_i^* : L_{-i} \longrightarrow \mathbb{R}$ is defined by

$$\overline{V}_i^*(s_{-i}) = \sup_{s_i \in L_i} V_i^*(s_i, s_{-i}).$$

Interim Payoff Security

A function $u_i(x_i, x_{-i})$ is transfer lower semicontinuous in x_{-i} if for every $\varepsilon > 0$ and every (x_i, x_{-i}) there exist $\overline{x}_i \in X_i$ and a neighborhood $N(x_{-i})$ in X_{-i} such that

$$u_i(\overline{x}_i, x'_{-i}) > u_i(x_i, x_{-i}) - \varepsilon \text{ for every } x'_{-i} \in N(x_{-i})$$

Proposition 1. Each interim payoff function $V_i : B_i \times S_{-i} \times T_i \to \mathbb{R}$ is transfer lower semicontinuous in (s_{-i}, t_i).

Corollary Each interim value function $\overline{V}_i : S_{-i} \times T_i \to \mathbb{R}$ is lower semicontinuous.

Interim Best-Reply Correspondences

Proposition 2 In Γ, each interim value function $\overline{V}_i : S_{-i} \times T_i \longrightarrow \mathbb{R}$ is continuous.

Corollary. In Γ, for every fixed i, $s_{-i} \in L_{-i}$, and $\varepsilon > 0$, the correspondence $M_i^\varepsilon : S_{-i} \times T_i \twoheadrightarrow B_i$ defined by $M_i^\varepsilon(s_{-i}, t_i) = \{b_i \in B_i : V_i(b_i, s_{-i}; t_i) > \overline{V}_i(s_{-i}; t_i) - \varepsilon\}$ has the local intersection property.

LIP: for every (s_{-i}, t_i), there exist $\bar{b}_i \in B_i$ and some $N(s_{-i}, t_i)$ such that $\bar{b}_i \in M_i^\varepsilon(s'_{-i}, t'_i)$ for all $(s'_{-i}, t'_i) \in N(s_{-i}, t_i)$

Tieless Single-Crossing Property

Definition. A Bayesian game $\Gamma = (B_i, T_i, f, u_i)_{i \in I}$ satisfies the upward TSCP (the downward TSCP) if for each $i \in I$, all b_{i1}, $b_{i2} \in B_i$ with $b_{i2} > b_{i1}$ (resp., $b_{i2} < b_{i1}$), and all nondecreasing strategies $s_j : T_j \to B_j$ of the other players $j \in I \backslash \{i\}$ such that $\mu(t_j \in T_j : b_{il} = s_j(t_j)) = 0$, $l = 1, 2$, the following condition holds:
If $V_i(b_{i2}, s_{-i}; t_i) - V_i(b_{i1}, s_{-i}; t_i) \geq 0$ for some $t_i \in T_i$, then the inequality holds when t_i rises (resp., falls).

Single-Valued Selections

Proposition 3. If Γ satisfies the TSCP, then, for every $i \in I$, $\varepsilon > 0$, and $s_{-i} \in S_{-i}$, the interim ε-best-reply correspondence $M_i^\varepsilon(s_{-i}, \cdot) : T_i \twoheadrightarrow B_i$ has a single-valued monotonic selection $s_i \in S_i$.

Recall: $M_i^\varepsilon(s_{-i}, t_i) = \{b_i \in B_i : V_i(b_i, s_{-i}; t_i) > \overline{V}_i(s_{-i}; t_i) - \varepsilon\}$

Consider trader i's ε-best-reply correspondence $\widetilde{M}_i^\varepsilon : S_{-i} \twoheadrightarrow S_i$ consisting of bidder i's interim ε-best replies and defined by $\widetilde{M}_i^\varepsilon(s_{-i}) = \{s_i \in S_i : V_i(s_i(t_i), s_{-i}; t_i) > \overline{V}_i(s_{-i}; t_i) - \varepsilon$ for almost all $t_i \in T_i\}$

Approximate Best-Reply Correspondences

Proposition 4. If Γ has the TSCP, then for every $\varepsilon > 0$, each ε-best-reply correspondence $\widetilde{M}_i^{\varepsilon} : S_{-i} \twoheadrightarrow S_i$ has the local intersection property.

Horvath's (1987) theorem:

Let $(X, \{F_A\})$ be a compact H-space, and let $M : X \twoheadrightarrow X$ be a correspondence with nonempty H-convex values that has the local intersection property. Then there exists $\overline{x} \in X$ such that $\overline{x} \in M(\overline{x})$.

Equilibrium Existence

Theorem 2 If Γ has the TSCP, then it possesses a monotone interim ε-equilibrium for every $\varepsilon > 0$.

Theorem 3. If Γ has the TSCP and $\sum_{i \in I} v_i(\cdot; t) : B \to \mathbb{R}$ is upper semicontinuous for every $t \in T$, then Γ has a Bayesian-Nash equilibrium.

Theorem 4. If Γ has the TSCP and each u_i is strictly increasing in own type, then Γ has a Bayesian-Nash equilibrium.

Example 1: A First-Price Auction with No PSBNE

Lebrun (1996)

(i) $T_1 = T_2 = [0, 1]$;

(ii) $B_1 = B_2 = [0, c]$, $c > 1$;

(iii) The bidders' types are independently uniformly distributed on $[0, 1]$;

(iv) $u_1(b_1; t) = -b_1$ and $u_2(b_2; t) = t_2 - b_2$ for all $(b, t) \in B \times T$.

Example 2. A First-Price Common-Value Auction

(i) n players;

(ii) $B_i = [0, c]$, $c > 1$; $T_i = [0, 1]$;

(iii) $u_i(b_i; t) = \max\{t_1, \ldots, t_n\} - b_i$ for all $(b_i, t) \in B_i \times T$

(iv) The bidders' types are independently uniformly distributed on $[0, 1]$.