Self-similar Attractors in Solow-type Public Debt Dynamics Generated by Iterated Function Systems on Density Functions

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Introduction

- We consider a Solow-type economic growth model describing the accumulation of public debt
- *Macroeconomic quantities* are *random variables* rather than deterministic amounts
- Specifically, we study the evolution through time of the *density functions* associated with such random variables
- Under appropriate contractivity conditions, we show that such dynamics generate self-similar objects that can be characterized as the fixed-point solution of an *Iterated Function Systems on Density Functions* (IFSDF)

Motivation

- Tackling public debt directly as a random variable, rather than a deterministic index, allows to take into account the uncertainty associated with the formation of expectations in modern economies in which the volatility of the cost of borrowing crucially determines the evolution of public debt
- The fixed-point solving the IFSDF is the *long-run distribution of the public debt*
- It depends on exogenous parameters as well as on policy tools (tax rate, public spending)
- Hence, the latter may be suitably chosen in order to affect its shape

Iterated Function Systems on Mappings (IFSM) I

Intuition

- Idea (Barnsley, 1989; Kunze et al., 2012): build a *fractal transform* operator $T : U \to U$ on an element u of the complete metric space (U, d) capable of
  1. producing a set of N spatially-contracted copies of u
  2. recombining them in order to get a new element $v \in U$, $v = Tu$
- Under appropriate conditions the transform T is a contraction and thus Banach's fixed point theorem guarantees the existence of a unique fixed point $\bar{u} = T\bar{u}$

- IFSMs (Forte and Vrscay, 1995) extend the classical notion of Iterated Function Systems (IFS) to the case of space of functions
- We consider the case of maps in $L^2([0,1])$: an IFSM can be used to approximate a given element u in such space
- Let

$$U = \left\{ u : [0,1] \to \mathbb{R}, u \in L^2([0,1]) \right\}$$

- The ingredients of an N-map IFSM on U are

1. a set of N *contractive mappings* $w = \{w_1, w_2, \dots, w_N\}$, $w_i(x) : [0,1] \to [0,1]$, often in affine form:

$$w_i(x) = s_i x + a_i, \qquad 0 \leq s_i < 1, \qquad i = 1, 2, \dots, N$$

2. a set of associated functions (*greyscale maps*) $\phi = \{\phi_1, \phi_2, \dots, \phi_N\}$, $\phi_i : \mathbb{R} \to \mathbb{R}$, again often affine:

$$\phi_i(y) = \alpha_i y + \beta_i$$

- Associated with the N-map IFSM (w, ϕ) is the *fractal transform* operator T defined as

$$(Tu)(x) = \sum_{i=1}^{N}{}' \phi_i \left(u \left(w_i^{-1} (x) \right) \right)$$

  - *'prime'* means the sum operates only on terms for which w_i^{-1} is defined

Proposition (Forte and Vrscay, 1995)

$T : U \to U$ and for any $u, v \in X$ we have

$$d(Tu, Tv) \leq C d(u, v), \qquad \text{where } C = \sum_{i=1}^{N} s_i^{\frac{1}{2}} |\alpha_i|$$

- When $C < 1$, T is contractive on U so that there exist a unique fixed point $\bar{u} \in U$ such that $\bar{u} = T\bar{u}$

Iterated Function Systems on Mappings (IFSM) IV

Interpretation

1. maps w_i, like in standard IFS, rescale the function u along the **horizontal axis**; for example the two maps

$$w_1(x) = (1/2)x, \qquad w_2(x) = (1/2)x + 1/2$$

transform the whole $[0,1]$ into $[1,1/2]$ and $[1/2,1]$ respectively

2. maps ϕ_i rescale the function u along the **vertical axis**; for example the two linear maps

$$\phi_1(y) \equiv py, \qquad \phi_2(y) \equiv (1-p)y, \qquad \text{with } 0 < p < 1,$$

together with $w_1(x) = (1/2)x$ and $w_2(x) = (1/2)x + 1/2$, contract the values of u by a factor (weight) of p over the sub-interval $[1,1/2]$ and by a factor (weight) of $1-p$ over the sub-interval $[1/2,1]$

3. T is a **purely deterministic** transform, randomness will be added later on when functions u will be interpreted as **density functions**

Iterated Function Density Functions (IFSDF) I

Definition

- The space of *density functions* is defined as

$$\bar{U} = \Big\{ u : [0,1] \to \mathbb{R} \text{ such that}$$

$$u \in L^2\left([0,1]\right), u\left(x\right) \geq 0 \ \forall x \in [0,1], \int_{[0,1]} u\left(x\right) \nu\left(dx\right) = 1 \Big\}$$

- ν is an arbitrary probability measure on $[0,1]$ and the space L^2 is defined with respect to ν (think of ν as *Lebesgue measure*)

Iterated Function Density Functions (IFSDF) II

Main result

Proposition

1. *The space $\bar{U}$ is complete with respect to the usual L^2 norm*
2. *Suppose that the following conditions are satisfied:*

   i) $\alpha_i, \beta_i \in \mathbb{R}_+$ *for all* $i = 1 \ldots N$
   ii) $\sum_{i=1}^{N} s_i \left(\alpha_i + \beta_i\right) = 1$

   then the operator T defined as $\left(Tu\right)\left(x\right) = \sum_{i=1}^{N}{}' \phi_i \left(u\left(w_i^{-1}\left(x\right)\right)\right)$ *maps $\bar{U}$ into itself.*
3. *Furthermore, if*

   iii) $\sum_{i=1}^{N} s_i^{\frac{1}{2}} \alpha_i < 1$

   then T is a contraction over $\bar{U}$, so that T has a unique fixed point that is a global attractor for any sequence of the form $u_{t+1} = Tu_t$ for any initial condition $u_0 \in \bar{U}$.

Solow-type Growth Model with Debt Accumulation I

Deterministic dynamics

- Small open economy, exogenous interest rate of international borrowing
- Public debt used to finance public spending
- Households consume all disposable income: $C_t = (1 - \tau) Y_t$
  - C_t consumption, Y_t income, $0 < \tau < 1$ tax rate
- Tax revenue $R_t = \tau Y_t$ entirely devoted to repay public debt
- Income grows exogenously at the rate $\gamma > 0$: $Y_{t+1} = (1 + \gamma) Y_t$
- Public spending = exogenous share, $0 < g < 1$, of income: $G_t = gY_t$
- G_t entirely financed via debt accumulation
- Exogenous Interest rate $r > 0$, interest payments: rB_t
  - B_t public debt
- Public debt accumulation dynamics:

$$B_{t+1} = (1 + r) B_t + G_t - R_t = (1 + r) B_t + gY_t - \tau Y_t$$

Solow-type Growth Model with Debt Accumulation II

- **Assumptions:**
  1. (La Torre and Marsiglio, 2019): tax rate τ a *linear function of debt-to-GDP ratio*: $\tau\left(\frac{B_t}{Y_t}\right) = \tau\frac{B_t}{Y_t}$, $\tau > 0$
  2. $0 \leq B_t \leq Y_t$, so that $x_t = \frac{B_t}{Y_t} \in [0, 1]$ for all t
- Then, *law of motion of the debt to GDP ratio*, $x_t = \frac{B_t}{Y_t}$:

$$x_{t+1} = \frac{1 + r - \tau}{1 + \gamma} x_t + \frac{g}{1 + \gamma}$$

  - a higher γ reduces the accumulation of the debt ratio by increasing resources to debt repayment
  - a higher interest rate increases the accumulation of the debt ratio by increasing interest payments
  - a higher income share of public spending increases the accumulation of the debt ratio by worsening the public budget balance position
  - a higher tax coefficient reduces the accumulation of the debt ratio by improving the public budget balance position
- If public budget balance in equilibrium, $G_t = R_t$, evolution of public debt would depend only on the gap between r and γ

Solow-type Growth Model with Debt Accumulation III

Stochastic dynamics

- Now x_t no longer a deterministic variable but a *random variable* with associated *density function* $u_t \in \bar{U}$
- Evolution of the density of the ratio variable x_t:

$$u_{t+1} = Tu_t = \sum_{i=1}^{N} p_i \left[\left(\frac{1 + r_i - \tau_i}{1 + \gamma_i} \right) u_t \cdot w_i^{-1} + \frac{g_i}{1 + \gamma_i} \right]$$

- $p_i \in [0, 1]$, $\sum_{i=1}^{N} p_i = 1$, probabilities associated with $i = 1, \ldots, N$ different *economic scenarios*, each characterized by different r_i interest rates on borrowing, γ_i growth rates of output, τ_i tax rates, g_i public spending share of GDP, $w_i : [0, 1] \to [0, 1]$ contractions
- Density of the level of the debt ratio at time $t + 1$, u_{t+1}, obtained by combining modified copies of the previous density at time t
- Each copy *vertically rescaled* by a combination of parameters p_i, r_i, τ_i, γ_i and g_i and *horizontally shifted* towards higher or lower debt ratio levels, x_{t+1}, by the composition with w_i^{-1}

- Our former Proposition can be applied with

$$\alpha_i = p_i \left(\frac{1 + r_i - \tau_i}{1 + \gamma_i} \right) \qquad \beta_i = p_i \frac{g_i}{1 + \gamma_i}$$

Proposition

If

i) $\sum_{i=1}^{N} s_i p_i \left[\frac{(1+r_i-\tau_i)+g_i}{1+\gamma_i} \right] = 1$

ii) $\sum_{i=1}^{N} s_i^{\frac{1}{2}} p_i \left(\frac{1+r_i-\tau_i}{1+\gamma_i} \right) < 1$

then the dynamic $u_{t+1} = \sum_{i=1}^{N} p_i \left[\left(\frac{1+r_i-\tau_i}{1+\gamma_i} \right) u_t \cdot w_i^{-1} + \frac{g_i}{1+\gamma_i} \right]$ *has a* **unique steady-state** $\bar{u}$ *that is globally attractive, i.e.,* $u_n \underset{L^2}{\rightarrow} \bar{u}$ *for any initial density* $u_0 \in \bar{U}$ *and is characterized by the following expression:*

$$\bar{u} = \sum_{i=1}^{N} p_i \left[\left(\frac{1 + r_i - \tau_i}{1 + \gamma_i} \right) \bar{u} \cdot w_i^{-1} + \frac{g_i}{1 + \gamma_i} \right]$$

$\bar{u}$ *is a* **self-similar** *object as it is the sum of distorted copies of itself*

Solow-type Growth Model with Debt Accumulation V

Interpretation

- In our model the rescaling along the **vertical axis** through the maps

$$\phi_i(y) = \alpha_i y + \beta_i = \left[p_i \left(\frac{1 + r_i - \tau_i}{1 + \gamma_i} \right) \right] y + p_i \frac{g_i}{1 + \gamma_i}$$

  takes into account all parameters' values, not only the probabilities p_i

- Specifically, the probability that the debt to GDP ratio at $t+1$ lies in a given values range,

$$Prob\left(a \leq x_{t+1} = \frac{B_{t+1}}{Y_{t+1}} \leq b \right) = \int_a^b u_{t+1}(x_{t+1})\, \nu(dx_{t+1}),$$

  depends on:

  1. the probabilities p_i of each scenario i
  2. the interest rates of international borrowing r_i in each scenario
  3. the tax parameters τ_i applied in each scenario
  4. the growth rates γ_i in each scenario
  5. the shares of income g_i for public spending in each scenario

An algorithm to approximate the invariant measure

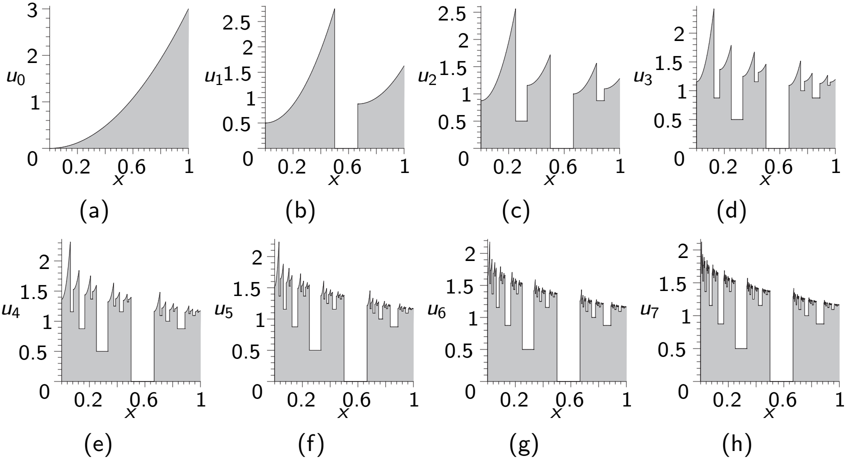
- Under the assumptions of the last Proposition there is a **unique density** $\bar{u}$ to which any initial density u_0 converges through iterations of the fractal operator

$$u_{t+1} = Tu_t = \sum_{i=1}^{N} \left(\alpha_i u_t \cdot w_i^{-1} + \beta_i\right)$$
$$= \sum_{i=1}^{N} \left[p_i \left(\frac{1 + r_i - \tau_i}{1 + \gamma_i} \right) u_t \cdot w_i^{-1} + p_i \frac{g_i}{1 + \gamma_i} \right]$$

- We exploit the “piecewise” routine embedded in Maple to build a simple algorithm that directly iterates the definition of operator T above transforming **any density** u_t into its next step density u_{t+1}
- No need to keep track of all intervals in each pre-fractal (the piecewise function routine in Maple does it automatically)
- No need to start from a simple initial density like the uniform density $u_0 \equiv 1$ (however its use speeds up the process)

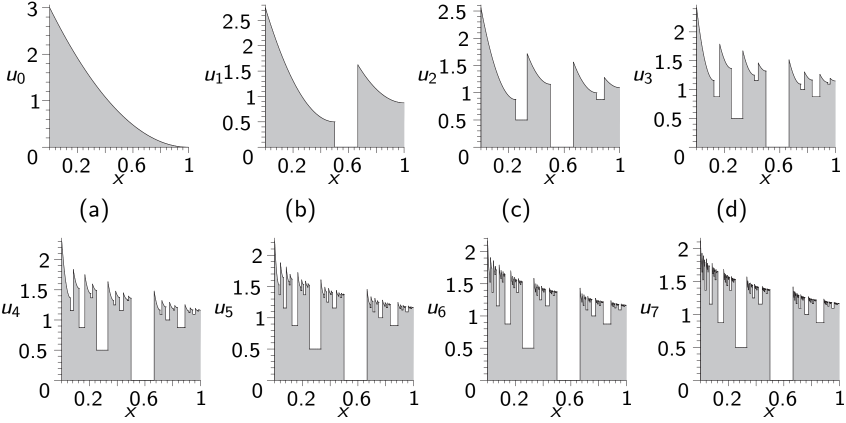
Examples with different initial densities I

- $N = 2$ no overlapping maps: $w_1(x) = \frac{1}{2}x$, $w_1(x) = \frac{1}{3}x + \frac{2}{3}$, $\phi_1(y) = \frac{3}{4}y + \frac{1}{2}$, $\phi_2(y) = \frac{1}{4}y + \frac{7}{8}$ satisfying conditions i) and ii) of last Proposition; first 7 iterations starting from $u_0 = 3x^2$:



Examples with different initial densities II

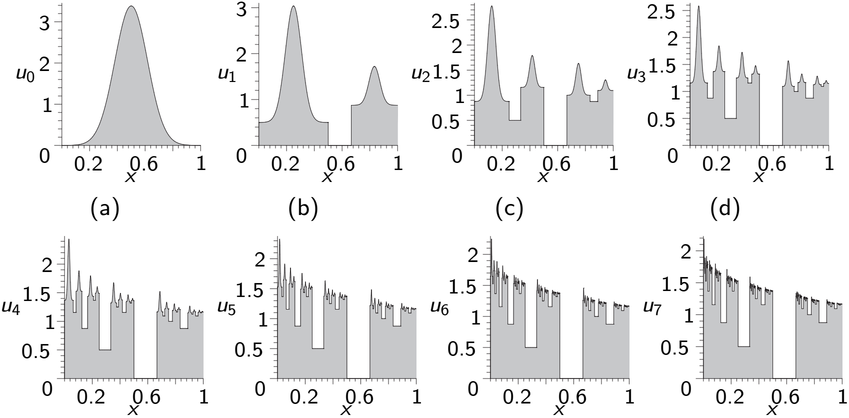
- Same parameters as before; first 7 iterations starting from $u_0 = 3(x-1)^2$:



(a) (b) (c) (d) (e) (f) (g) (h)

Examples with different initial densities III

- Same parameters as before; first 7 iterations starting from (bell-shaped) $u_0 = 3,385e^{-(6x-3)^2}$:

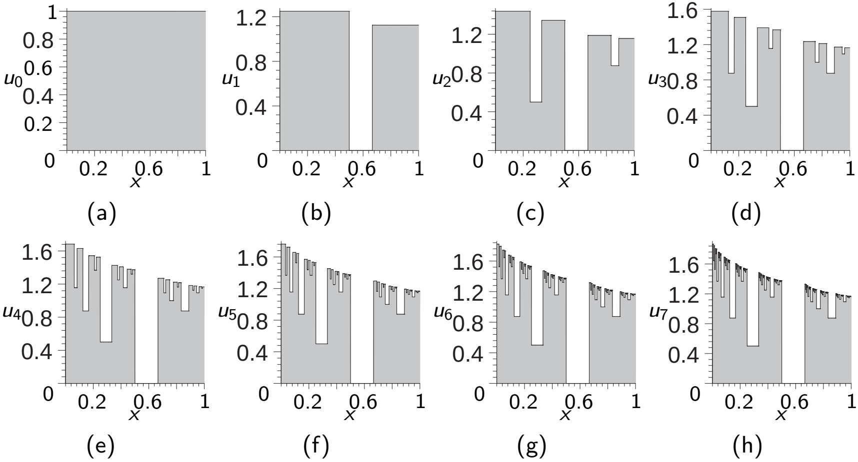


(a) (b) (c) (d)

(e) (f) (g) (h)

Examples with different initial densities IV

- Same parameters as before; first 7 iterations starting from (uniform) $u_0 \equiv 1$:



Examples with different initial densities V

Comments

- Any specific features of the initial density u_0 are wiped out by
  1. the properties of the **horizontal rescaling** introduced by the maps w_i (e.g., the 'holes' when images of the w_i do not overlap)
  2. the values of parameters

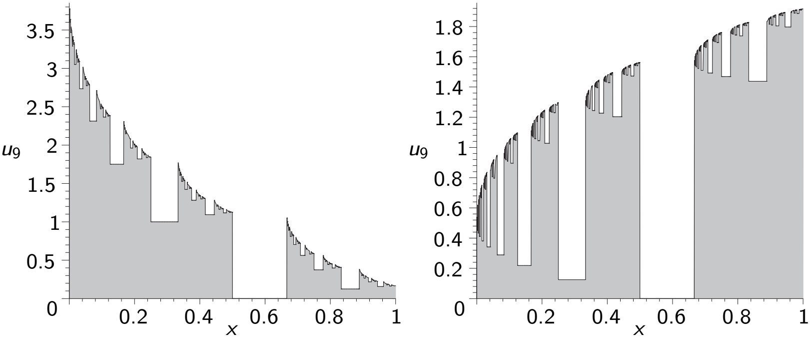
$$\alpha_i = p_i \left(\frac{1 + r_i - \tau_i}{1 + \gamma_i} \right) \qquad \text{and} \qquad \beta_i = p_i \frac{g_i}{1 + \gamma_i}$$

  defining the *greyscale maps* $\phi_i(y) = \alpha_i y + \beta_i$, which determine the **vertical rescaling** of the marginal density u_t introduced after each iteration

- Unlike standard IFS on variables, after the first iteration parameters β_i, when are positive, add always a positive value to the marginal density u_t, also on the 'holes' of each prefractal generated by the maps w_i when their images do not overlap.

Other non-overlapping examples

- Same parameters as before except for the β_is; 9^{th} iteration starting from $u_0 \equiv 1$:

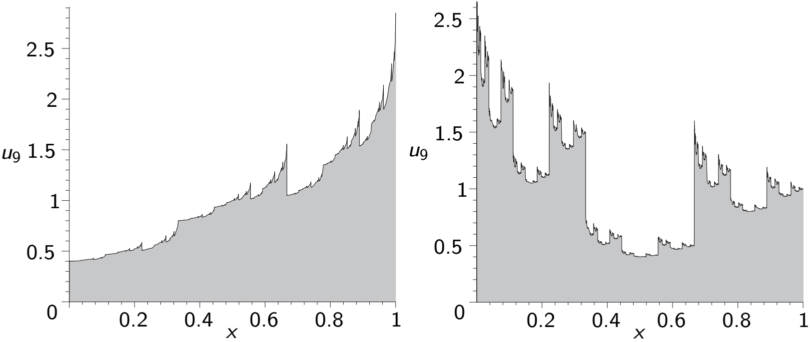


(a) $\beta_1 = 1,\ \beta_2 = \frac{1}{8}$ (b) $\beta_1 = \frac{1}{8},\ \beta_2 = \frac{23}{16}$

- **Parameters $\boldsymbol{\beta_i}$s** are actually crucial in establishing **"generic" monotonicity** properties of the invariant density

Examples with wavelets

- $N = 3$ maps with almost overlapping images: $w_1(x) = \frac{1}{3}x$, $w_2(x) = \frac{1}{3}x + \frac{1}{3}$, $w_3(x) = \frac{1}{3}x + \frac{2}{3}$, ϕ_is satisfying conditions i) and ii) of last Proposition; 9^{th} iteration starting from $u_0 \equiv 1$:

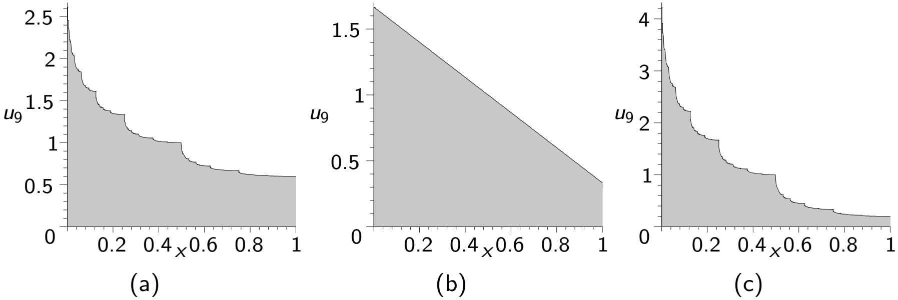


(a) $\phi_1(y) = \frac{1}{6}y + \frac{1}{3}$, $\phi_2(y) = \frac{1}{3}y + \frac{2}{3}$, $\phi_3(y) = \frac{3}{4}y + \frac{3}{4}$

(b) $\phi_1(y) = \frac{3}{4}y + \frac{3}{4}$, $\phi_2(y) = \frac{1}{6}y + \frac{1}{3}$, $\phi_3(y) = \frac{1}{3}y + \frac{2}{3}$

The role of greyscale maps' parameters in monotonicity

- $N = 2$ wavelets maps: $w_1(x) = \frac{1}{2}x$, $w_2(x) = \frac{1}{2}x + \frac{1}{2}$
- *Decreasing* values of either α_i or β_i (both reinforce each other) determine a *decreasing* limiting density; 9^{th} iteration from $u_0 \equiv 1$:



(a) $\alpha_1 = \frac{5}{6}$, $\alpha_2 = \frac{1}{6}$, $\beta_1 = \beta_2 = \frac{1}{2}$: $\alpha_1 > \alpha_2$ and β_is neutral

(b) $\alpha_1 = \alpha_2 = \frac{1}{2}$, $\beta_1 = \frac{5}{6}$, $\beta_2 = \frac{1}{6}$: α_is neutral and $\beta_1 > \beta_2$

(c) $\alpha_1 = \beta_1 = \frac{5}{6}$, $\alpha_2 = \beta_2 = \frac{1}{6}$: both $\alpha_1 > \alpha_2$ and $\beta_1 > \beta_2$

Conclusions

$\subset$

- We apply the theory on *Iterated Function Systems on Density Functions* (IFSDF) to a Solow-type economic growth model describing the accumulation of public debt and show that, under appropriate contractivity conditions, such dynamics converge to a unique long-run density
- Parameters $\alpha_i = p_i \left(\frac{1+r_i-\tau_i}{1+\gamma_i} \right)$ and $\beta_i = p_i \frac{g_i}{1+\gamma_i}$ of the *greyscale maps* $\phi_i(y) = \alpha_i y + \beta_i$, which determines the **vertical rescaling** of the marginal density u_t through the Fractal operator, establish the essential features of the long-run density
- As coefficients α_i and β_i, besides depending on exogenous parameters (international interest rates r_i, growth rates γ_i, and probabilities p_i), depend on **policy parameters** like tax rates τ_i and public spending g_i, the latter can be suitably chosen so to affect the *long-run density of the debt-to-GDP ratio* $x_t = \frac{B_t}{Y_t}$

Further research

1. Use the Collage Theorem to approximate the (policy) parameters τ_i and g_i in order to build a Fractal operator capable of generating an IFSDF converging to any target invariant density
   - for example a decreasing limiting density that concentrates most of the debt values closer to the 0 endpoint of $[0, 1]$
2. Extend optimization techniques (e.g., the Calculus of Variations) to intertemporal stochastic problems having (possibly fractal) densities rather than real variables as states and controls

THANK YOU!