Modelling behavioral change in a peer-driven consumer network

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Background

- Literature on endogenous preference change Benhabib and Day (1981)
- Consumption as a social activity Gaertner and Jungeilges (1988), Gaertner and Jungeilges (1993)
- Emphasis on non-market aspects
- Dominance of deterministic arguments
- Rigorous re-analysis: Ekaterinchuk et al. (2017a), Ekaterinchuk et al. (2017b)
- Stochastic model without interaction : Jungeilges et al. (2018), Jungeilges and Ryazanova (2019)

Scope of the study

Object: A stochastic consumption model with interacting agents

- focus on special case: peer driven
- area of coexisting attractors
- types of noise: additive and parametric

Goal: Focussing on the case of increasing influence of one individual on another, analyse the transition between coexisting attractors closed (closed invariant curves and k-cycles)

- Identify different types of transitions.
- Unravel the "genesis" of the transitions.

Method: Mixed method approach

- Indirect method of studying stochastic dynamics
- Stochastic sensitivity function technique (SSF) Milstein and Ryashko (1995)
- Numerical tools Panchuk (2015)

The Model

$$x_{t+1} = f(x_t) + \varepsilon g(x_t)\xi_t, \tag{1}$$

where f represents the 2D noninvertible map $f : \mathbb{R}^2_+ \to \mathbb{R}^2_+$

$$f(x_t) = \begin{pmatrix} \frac{b_1}{p_x p_y}\left(\alpha_1 x_{1t}(b_1 - p_x x_{1t}) + D_{12} x_{2t}(b_2 - p_x x_{2t})\right) \\ \\ \frac{b_2}{p_x p_y}\left(\alpha_2 x_{2t}(b_2 - p_x x_{2t}) + D_{21} x_{1t}(b_1 - p_x x_{1t})\right) \end{pmatrix} \tag{2}$$

and g denotes the smooth matrix function

$$g(x_t) = \begin{pmatrix} \iota_1 \frac{b_1 D_{12}}{p_x p_y} x_{2t} & \iota_2 & 0 \\ \\ 0 & 0 & \iota_3 \end{pmatrix}. \tag{3}$$

(i) $\iota = (0, 1, 1)$: additive noise

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Constraints on the parameter space

Feasible phase region

If $\alpha_1 b_1^2 + D_{12} b_2^2 < 4 p_x p_y, \quad \alpha_2 b_2^2 + D_{21} b_1^2 < 4 p_x p_y$ holds, then $f(S) \subset S$ where

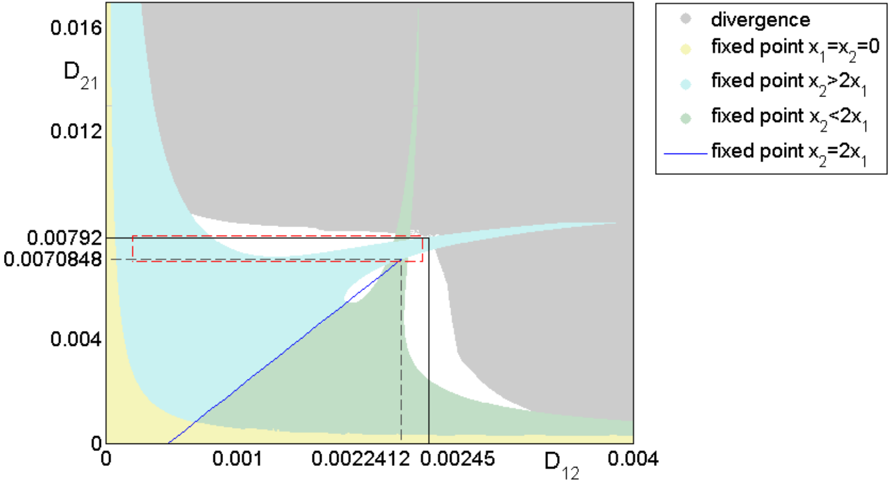
$$S = \left(0, \frac{b_1}{p_x}\right) \times \left(0, \frac{b_2}{p_x}\right) \tag{4}$$

is the feasible phase region.

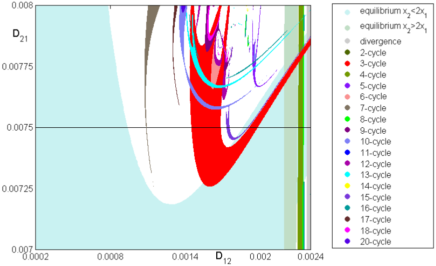
Economic environment: $p = (p_x, p_y) = \left(\frac{1}{4}, 1\right), \ b = (b_1, b_2) = (10, 20)$

1. $D_{12} < 0.25(0.01 - \alpha_1), D_{12} < 4(0.0025 - \alpha_2), \ S = (0, 40) \times (0, 80)$
2. Fix learning parameters $\alpha_1 = 0.0002, \alpha_2 = 0.00052$
3. $D = \{(D_{12}, D_{21}) \mid 0 \leq D_{12} \leq 0.004 \wedge 0 \leq D_{21} \leq 0.016\}$
4. $D^e = \{(D_{12}, D_{21}) \mid 0 \leq D_{12} \leq 0.00245 \wedge 0 \leq D_{21} \leq 0.00792\} \subseteq D,$

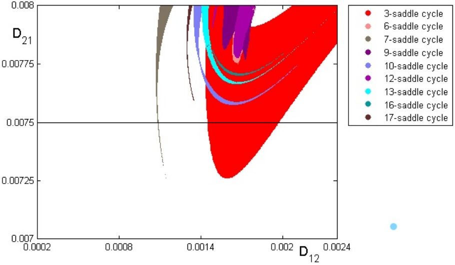
2D bifurcation diagram for D ($D^e \subset D$)



2D bifurcation diagrams

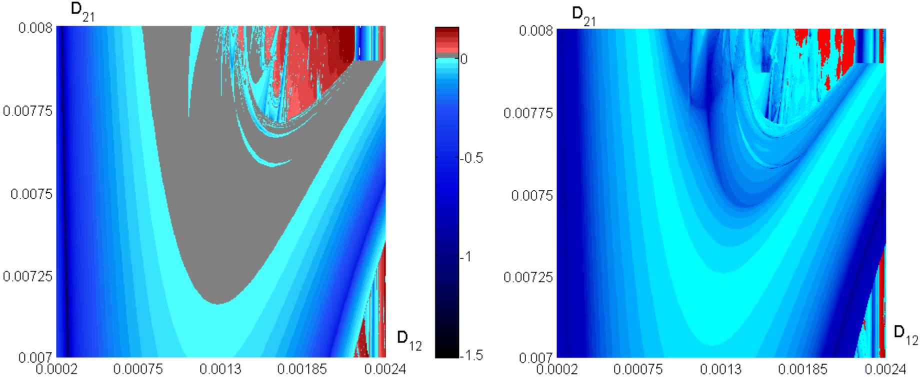


stable *k*-cycles



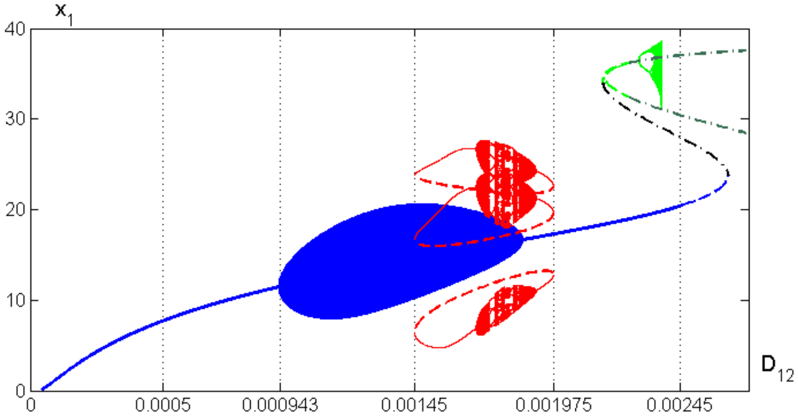
saddle *k*-cycles

Lyapunov exponents - super chaos

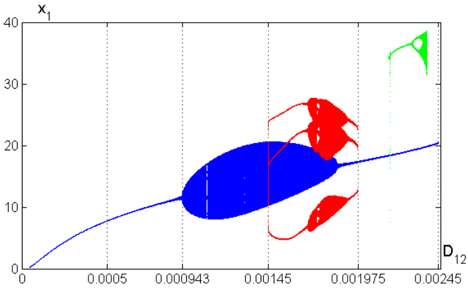


λ_{max} λ_{min}

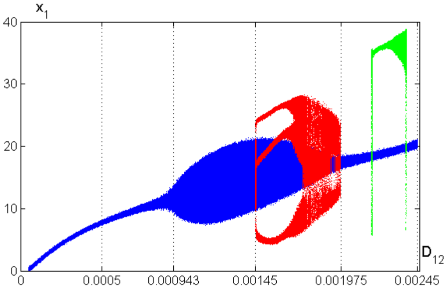
Bifurcation diagram $D_{21} = 0.0075$



Bifurcation diagram: stochastic case (additive)

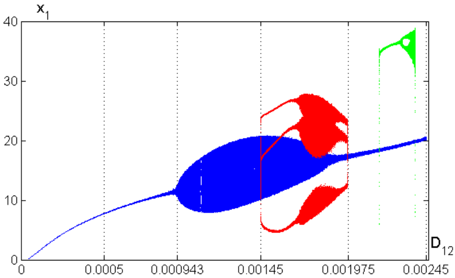


$\varepsilon = 0.01$



$\varepsilon = 0.1$

Bifurcation diagram: stochastic case (parametric)



$\varepsilon = 0.01$ $\varepsilon = 0.1$

Sensitivity analysis

Assumption X

The deterministic consumption system (3) posses a regular attractor.

Notation:

- x_t denotes a solution of the deterministic system (3)
- $X_t \equiv x_t(\varepsilon)$ solution of the stochastic system (1)

Asymptotics

Let $\Delta_t(\varepsilon) = x_t(\varepsilon) - \gamma$ then $z_t = \lim_{\varepsilon \to 0} \frac{\Delta_t(\varepsilon)}{\varepsilon}$.

Interpretation: For small ε, $V_t = \mathbb{E}[z_t z_t^\top]$ estimates the dispersion of random states around γ.

Sensitivity of a steady state ($\gamma = \bar{x}$)

Basic idea: Sensitivity of a steady state captures the variation of the stochastic trajectory X_t around the stable steady state $\bar{x}$ of the deterministic skeleton.

Sensitivity matrix

The sensitivity matrix W associated with $\bar{x}$ solves $W = FWF' + Q$ where $F = \frac{\partial f}{\partial x}(\bar{x})$ and $Q = g(\bar{x})g(\bar{x})'$.

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Interpretation: $W \approx$ covariance matrix of states

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Density of states

A Gaussian approximation of the density of states $p(x, \varepsilon)$ based on the eigenvalues $\lambda_1, \ldots, \lambda_p$ and eigenvectors of W can be given.

Spin off: Confidence ellipse around $\bar{x}$

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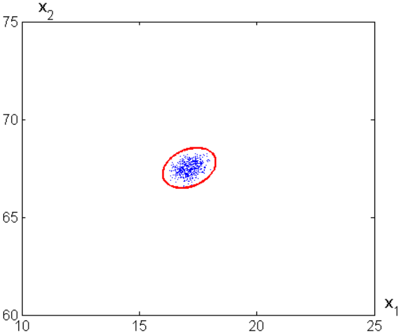
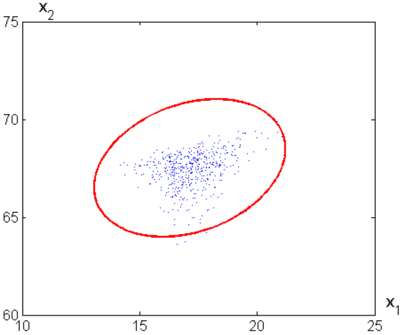
Steady state: confidence ellipse



additive $\varepsilon = 0.01$



parametric $\varepsilon = 0.1$

Sensitivity of a k-cycle ($\gamma = C_k$)

Cycle

Let $C_k = \{\bar{x}_1, \ldots, \bar{x}_k\}$ denote a stable k cycle such that $\bar{x}_{i+1} = f(\bar{x}_i)$ for $i = 1, \ldots, k-1$ and $\bar{x}_1 = f(\bar{x}_k)$.

Sensitivity matrices for the k-cycle

Let W_i denote the sensitivity matrix for the i'th element of a k-cycle. The elements of $\{W_1, W_2, \ldots, W_k\}$ are obtained as: $W_1 = BW_1B' + Q$ and

- for $i = 1, 2, \ldots, k-1$: $W_{i+1} = F_i W_i F_i' + Q_i$ with $F_i = \frac{\partial f}{\partial x}(\bar{x}_i)$, $Q_i = g(\bar{x}_i)g(\bar{x}_i)'$
- $B = \prod_{i=1}^{k} F_i$ and $Q = Q_k + F_k Q_{k-1} F_k' + \ldots + F_k \ldots F_2 Q_1 F_2' \ldots F_k'$

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Sensitivity function for the k-cycle

The eigenvalues $\lambda_{i1}, \ldots, \lambda_{ip}$ of each W_i, $\forall i$, quantify the sensitivity of the k-cycle.

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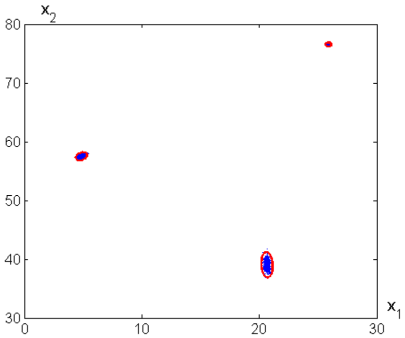
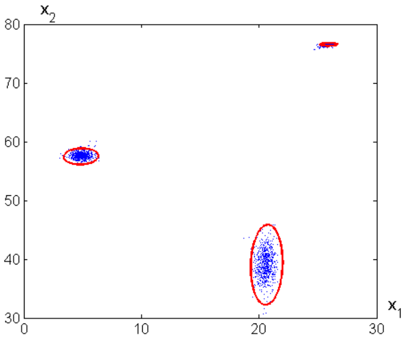
Confidence sets for a 3-cycle, $\varepsilon = 0.1$



additive



parametric

Sensitivity of a closed invariant curve (CIC)

The attractor of the deterministic system (3) is a closed invariant curve γ.

Definition 1

Let Π_t be the hyperplane orthogonal to γ at point ξ_t. P_t denotes the associated projection matrix (onto Π_t).

The dynamics takes the form

$$z_{t+1} = P_{t+1}[F_t z_t + g_t \xi_t]$$

where $F_t = \frac{\partial f}{\partial x}(\bar{x}_t)$ and $g_t = g(\bar{x}_t)$

Result 1

$V_t = \mathbb{E}[z_t z_t^\top]$ satisfies

$$V_{t+1} = P_{t+1}[F_t V_t F_t^\top + G_t] P_{t+1} \quad (5)$$

with $G_t = g_t g_t^\top$.

SSF closed invariant curve: Periodic case

- γ consists of a set of k cycles.
- γ exponentially stable $\Rightarrow$ (5) has a stable k periodic solution M_t such that $\lim_{t\to\infty}(M_t - V_t) = 0$

Determination of M_ts

$M_1 = P_1[\Phi M_1 \Phi^\top]P_1$ where $\Phi = F_k P_k F_{k-1} \cdots p_2 F_1$ and $Q = Q^{(k)}$

$$Q^{(0)} = 0 \tag{6}$$

$$Q^{(j)} = P_{j+1}[F_j Q^{(j-1)} F_j^\top + G_j]P_{j+1} \quad j = 1, 2, \ldots, k-1 \tag{7}$$

$$Q^{(k)} = F_k Q^{(k-1)} F_k^\top + G_k \tag{8}$$

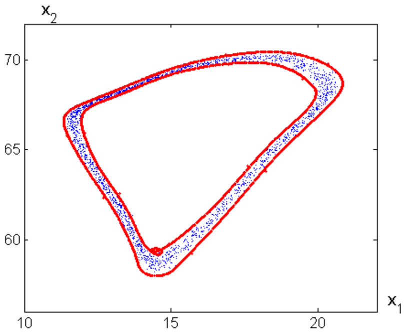
$M_{t+1} = P_{t+1}[F_t M_t F_t^\top + G_t]P_{t+1}$

Interpretation: $\{M_1, M_2, \ldots, M_k\}$ characterize SF of the k-cycle. For any $x_i \in \gamma$ $M_i \approx \varepsilon^2 M_i$ covariance matrix of states in the hyperplane Π_t orthogonal to γ at point $\bar{x}_i$.

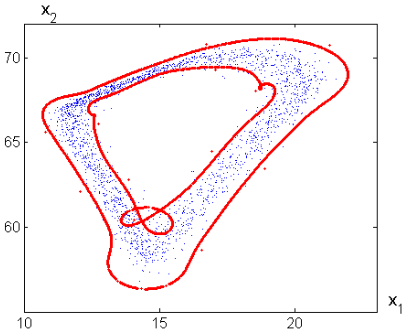
Closed invariant curve: Quasiperiodic case

- $\bar{x} \in \gamma$ and consider the solution x_t with $x_0 = \bar{x}$
- Points of the solution lie everywhere dense on γ.
- For any $\delta > 0$ there exists a k such that $\mid \bar{x}_{k+1} - \bar{x}_1 \mid < \delta$.
- Consider the points $\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k\}$ as elements of a k-cycle as a δ approximation of the initial quasiperiodic solution.
- Use method devised for the *periodic case* to obtain the SSF for γ.

CIC with confidence band, $\varepsilon = 0.05$

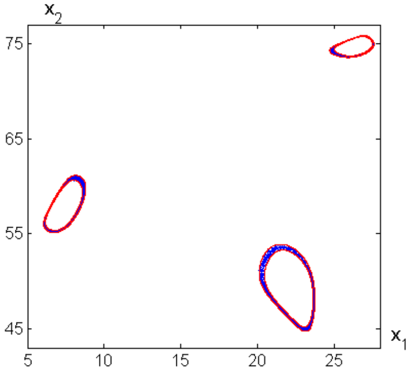


additive



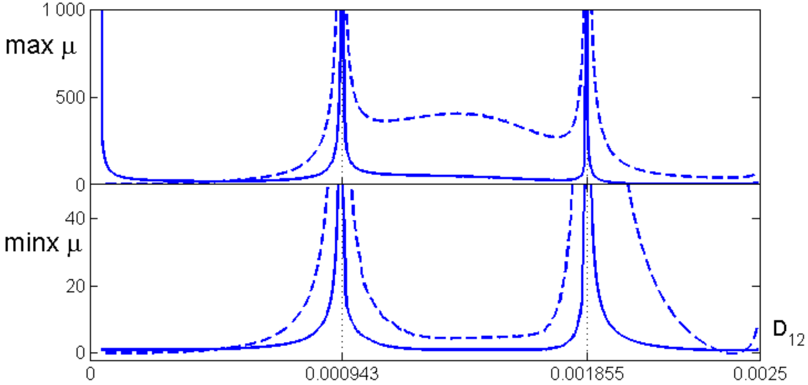
parametric

Confidence bands for 3-part CIC, $\varepsilon = 0.05$



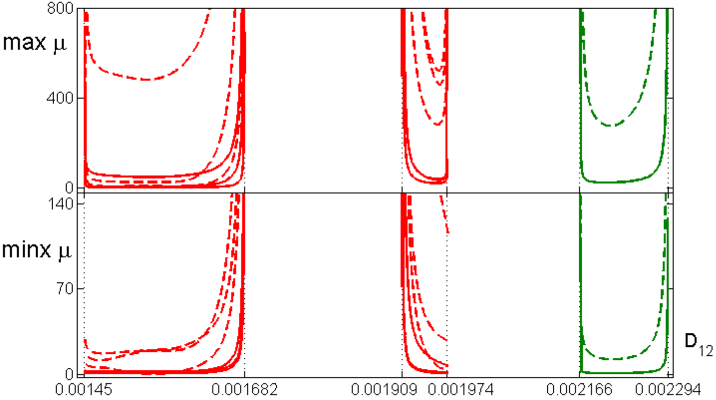
additive *parametric*

Sensitivity function for the "spindle"



$D_{21} = 0.0075$

SSF for 3-cycles



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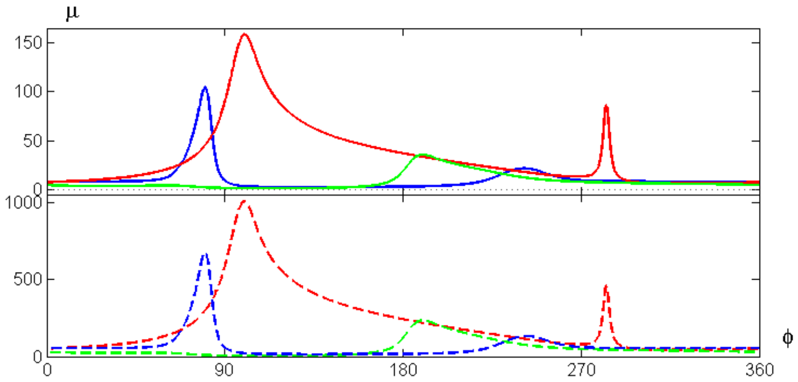
SSF for 3-part CIC



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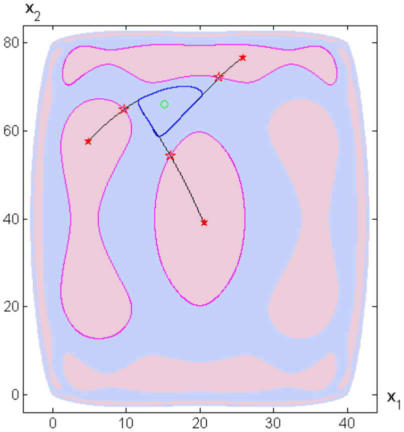
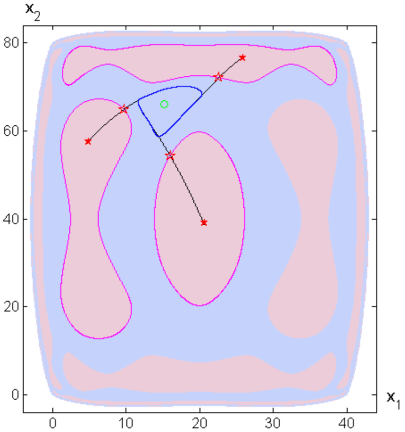
Coexistence: CIC and 3-cycle (total)



$D_{21} = 0.0075$, $D_{12} = 0.00157$

Thank you CompDTIMe

Coexistence: CIC and 3-cycle (total)



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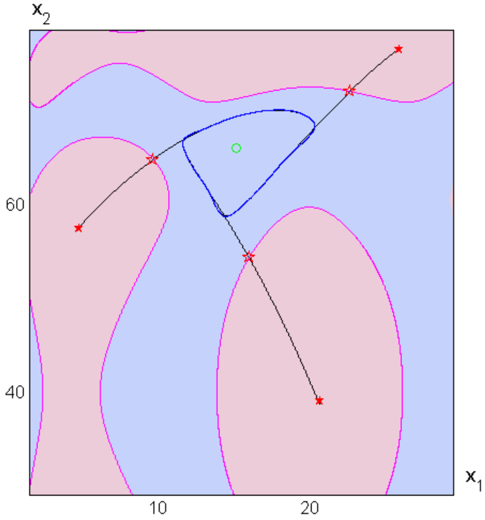
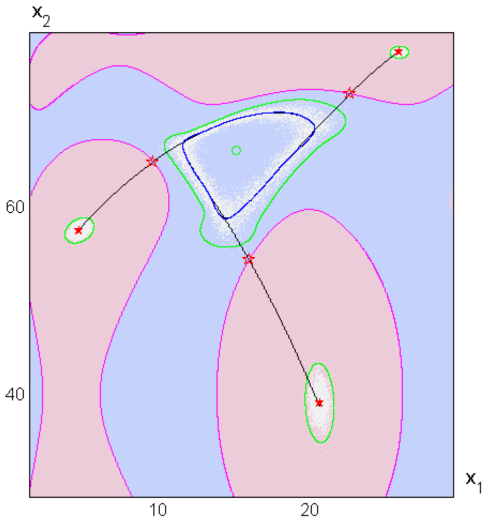
Thank you CompDTIMe

Coexistence: CIC and 3-cycle (zoom)



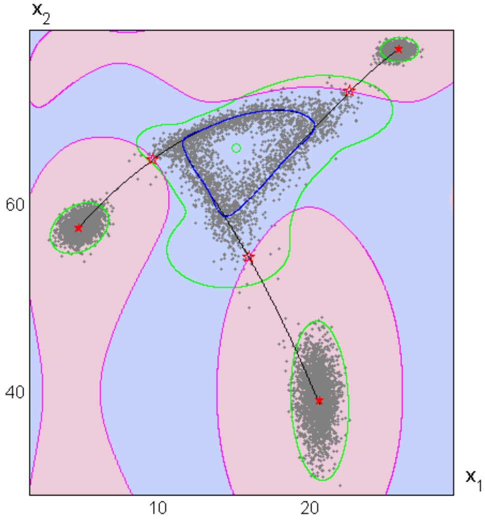
$D_{21} = 0.0075$, $D_{12} = 0.00157$

Coexistence: CIC and 3-cycle, additive noise $\varepsilon = 0.2$



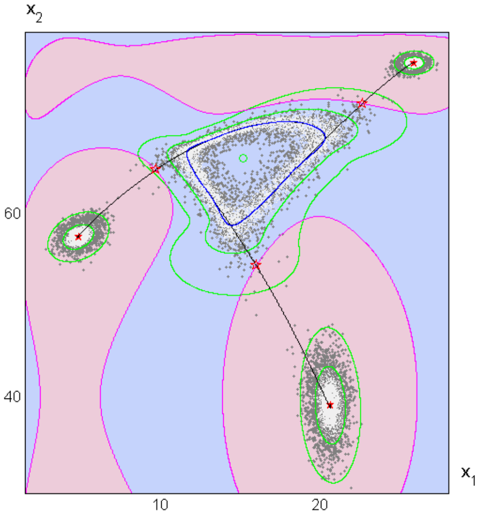
$D_{21} = 0.0075$, $D_{12} = 0.00157$

Transition: CIC $\mapsto$ 3-cycle, additive noise, $\varepsilon = 0.5$



$D_{21} = 0.0075$, $D_{12} = 0.00157$

Transition: CIC $\mapsto$ 3-cycle, additive noise, $\varepsilon = 0.2, \varepsilon = 0.5$



$D_{21} = 0.0075, D_{12} = 0.00157$

Conclusion

- Types of transitions: CIC $\mapsto$ 3-cycle
- Key elements in the genesis of transition:
  1. location of the steady state or cycle elements in their respective basins of attraction,
  2. sensitivity of the attractor as reflected in respective confidence ellipses.
- Transitions are more likely to occur under parametric noise than under additive noise.
- The noise levels at which transitions become likely depends on the level of influence.
- The unstable manifold of the saddle *k*-cycle plays a significant role for the transition process.
- Unstable manifolds should be considered in the modelling of behavioral transition.

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