Time Delays and Chaos in Two Competitng Species Revisited

Akio Matsumoto, Ferenc Szidarovszky

11th NED, 3 September, 2019
KSE, Kyiv

Motivation

- Shibata, A., and Saito, N., (1980): Time Delays and Chaos in Two Competition Species, *Mathematical Biosciences, 51, 313-325.*

Motivation

- Shibata, A., and Saito, N., (1980): Time Delays and Chaos in Two Competition Species, *Mathematical Biosciences, 51, 313-325.*
- It numerically illustrates that a Lotka-Volterra competition system with two discrete time delays can generate rich dynamics including chaos,

$$\dot{x}(t) = x(t)\left[\varepsilon_1 - a_{11}x(t-\tau_x) - a_{12}y(t)\right]$$

$$\dot{y}(t) = y(t)\left[\varepsilon_2 - a_{21}x(t) - a_{22}y(t-\tau_y)\right]$$

Motivation

- Shibata, A., and Saito, N., (1980): Time Delays and Chaos in Two Competition Species, *Mathematical Biosciences, 51, 313-325.*
- It numerically illustrates that a Lotka-Volterra competition system with two discrete time delays can generate rich dynamics including chaos,

$$\dot{x}(t) = x(t)\left[\varepsilon_1 - a_{11}x(t-\tau_x) - a_{12}y(t)\right]$$

$$\dot{y}(t) = y(t)\left[\varepsilon_2 - a_{21}x(t) - a_{22}y(t-\tau_y)\right]$$

- x and y are the population densities of two competitive species

Motivation

- Shibata, A., and Saito, N., (1980): Time Delays and Chaos in Two Competition Species, *Mathematical Biosciences, 51, 313-325.*
- It numerically illustrates that a Lotka-Volterra competition system with two discrete time delays can generate rich dynamics including chaos,

$$\dot{x}(t) = x(t)\left[\varepsilon_1 - a_{11}x(t-\tau_x) - a_{12}y(t)\right]$$

$$\dot{y}(t) = y(t)\left[\varepsilon_2 - a_{21}x(t) - a_{22}y(t-\tau_y)\right]$$

- x and y are the population densities of two competitive species
- $\varepsilon_i > 0$: intrinsic growth rate, $a_{ij} > 0$ $i, j = 1, 2$.

Motivation

- Shibata, A., and Saito, N., (1980): Time Delays and Chaos in Two Competition Species, *Mathematical Biosciences, 51, 313-325.*
- It numerically illustrates that a Lotka-Volterra competition system with two discrete time delays can generate rich dynamics including chaos,

$$\dot{x}(t) = x(t)\left[\varepsilon_1 - a_{11}x(t-\tau_x) - a_{12}y(t)\right]$$

$$\dot{y}(t) = y(t)\left[\varepsilon_2 - a_{21}x(t) - a_{22}y(t-\tau_y)\right]$$

- x and y are the population densities of two competitive species
- $\varepsilon_i > 0$: intrinsic growth rate, $a_{ij} > 0$ $i, j = 1, 2$.
- $\tau_x > 0$ and $\tau_y > 0$ are maturation delays of species

Motivation

- Shibata-Saito (1980) show that the dynamic behavior depends on the time delays τ_1 and τ_2.

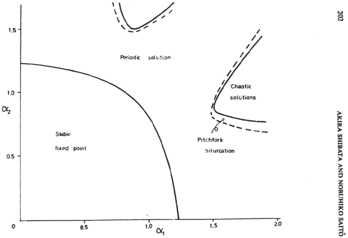


FIG. 1. Three types of attractors of the equations (4). The dynamical behaviors depend on the time lags α_1, α_2.

Motivation

- Shibata-Saito (1980) show that the dynamic behavior depends on the time delays τ_1 and τ_2.

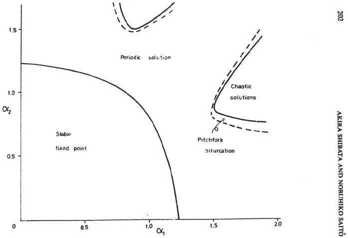


FIG. 1. Three types of attractors of the equations (4). The dynamical behaviors depend on the time lags α_1, α_2.

- Our purpose is to provide an analytical underpinning for their numerical simulations.

Motivation

- Fig3 of Shibata-Saito (1980) where $\tau_1 = 1.6$ and $\tau_2 = 0.85$

204 AKIRA SHIBATA AND NOBUHIKO SAITÔ

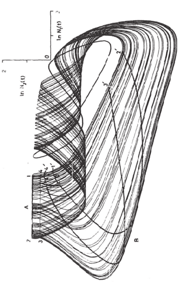


FIG. 3. Chaotic solutions of (4). $\alpha_1 = 1.6$, $\alpha_2 = 0.85$. The square A (1234) is folded to B (1'2'3'4' respectively from 1234) and the two are superposed. There is another periodic solution enclosing the chaotic solutions.

What we are going to do:

1. Introduction
2. Two delay Lotka-Volterra model (the stability switching curve)
3. Numerical simulations
4. Concluding Remarks

Introduction

The no-delay interaction between two species has three kinds of fundamental forms,

$$\dot{x}(t) = x(t)\left[\varepsilon_1 - a_{11}x(t) - a_{12}y(t)\right]$$

$$\dot{y}(t) = y(t)\left[\varepsilon_2 - a_{21}x(t) - a_{22}y(t)\right]$$

where a_{ii} is the strength of intra-competition within species i and a_{ij} is the strength of inter-competition between species i and j
(i) competition $a_{ii} > 0$ and $a_{ij} > 0$
(ii) cooperation: $a_{ii} > 0$ and $a_{ij} < 0$
(iii) prey-predation: $a_{11} > 0$, $a_{12} > 0$ and $a_{22} > 0$, $a_{21} < 0$; x-prey and y-predator

Introduction

- The steady state of the no-delay model are

$$x^e = \frac{\varepsilon_1 a_{22} - \varepsilon_2 a_{12}}{a_{11} a_{22} - a_{12} a_{21}} \text{ and } y^e = \frac{\varepsilon_2 a_{11} - \varepsilon_1 a_{21}}{a_{11} a_{22} - a_{12} a_{21}}$$

- Two species can coexist (i.e., $x^e > 0$ and $y^e > 0$) when the intra-competition dominates the inter-competition,

$$\frac{a_{11}}{a_{21}} > \frac{\varepsilon_1}{\varepsilon_2} > \frac{a_{12}}{a_{22}}.$$

Introduction: existing literature (competition system)

- Song, Y., Han, M., Pend, Y., (2004, *Chaos, Solitions and Fractals, 22, 1139-1148*)

$$\dot{x}(t) = x(t)\left[\varepsilon_1 - a_{11}x(t) - a_{12}y(t-\tau_x)\right]$$

$$\dot{y}(t) = y(t)\left[\varepsilon_2 - a_{21}x(t-\tau_y) - a_{22}y(t)\right]$$

- $\tau_x > 0$ and $\tau_y > 0$ are hunting delays.
- The hunting delays are harmless.

Introduction: existing literature (competition system)

- Zhang, J., Jin, Z., Yang, Y., Sun, G. (2009, *Nonlinear Analysis, 70, 849-860*)

$$\dot{x}(t) = x(t)\left[\varepsilon_1 - a_{11}x(t-\tau) - a_{12}y(t-\tau)\right]$$

$$\dot{y}(t) = y(t)\left[\varepsilon_2 - a_{21}x(t-\tau) - a_{22}y(t-\tau)\right]$$

- the muturation delays are equal to the hunding delays
- A hopf bifurcation arises.

Introduction: existing literature (competition system)

- Zhang, J., (2012, *Nonlinear Dynamics, 70, 849-800, 2012*)

$$\dot{x}(t) = x(t)\left[\varepsilon_1 - a_{11}x(t-\tau) - a_{12}y(t-\tau_x)\right]$$

$$\dot{y}(t) = y(t)\left[\varepsilon_2 - a_{21}x(t-\tau_y) - a_{22}y(t-\tau)\right]$$

- the maturation delays are the same but the hunting delays are different
- A hopf bifurcation occurs.

Two-delay Model

- The homogenous part of the Lotka-Volterra model with two delays,

$$\dot{x}(t) = -\alpha_x x(t - \tau_x) - \beta_x y(t)$$

$$\dot{y}(t) = -\beta_x x(t) - \alpha_y y(t - \tau_y)$$

with

$$\alpha_x = a_{11} x^e, \ \alpha_y = a_{22} y^e, \ \beta_x = a_{12} x^e, \ \beta_y = a_{21} y^e$$

Two-delay Model

- Substituting exponential solutions, $x(t) = e^{\lambda t} u$ and $y(t) = e^{\lambda t} v$ into the homogemous parts yields the corresponding characteristic equation,

$$\det \begin{pmatrix} \lambda + \alpha_x e^{-\lambda \tau_x} & \beta_x \\ \beta_y & \lambda + \alpha_y e^{-\lambda \tau y} \end{pmatrix} =$$

$$P_0(\lambda) + P_1(\lambda) e^{-\lambda \tau_x} + P_2(\lambda) e^{-\lambda \tau_y} + P_3(\lambda) e^{-\lambda(\tau_x + \tau_y)} = 0$$

wit

$$P_0(\lambda) = \lambda - \beta_x \beta_y, \;\; P_1(\lambda) = \alpha_x \lambda, \;\; P_2(\lambda) = \alpha_y \lambda, \;\; P_3(\lambda) = \alpha_x \alpha_y$$

Two-delay Model

- No delay case ($\tau_x = \tau_y = 0$), the characteristic equation is

$$\lambda^2 + (\alpha_x + \alpha_y)\lambda + (\alpha_x\alpha_y - \beta_x\beta_y) = 0$$

- Since $\alpha_x + \alpha_y > 0$ and $a_x\alpha_y - \beta_x\beta_y = (a_{11}a_{22} - a_{12}a_{21})x^e y^e > 0$,

the steady state is locally asymptotically stable
if $\tau_x = \tau_y = 0$

$$\Downarrow$$

Stability can be lost if $\tau_x > 0$ and $\tau_y > 0$ and How?

Two delay Model

- $\lambda = 0$ does not solve the characteristic eqation, $\alpha_x\alpha_y - \beta_x\beta_y \neq 0$
- We assume that $\lambda = i\omega$ with $\omega > 0$. The characteristic equation is changed to

$$P_0(i\omega) + P_1(i\omega)e^{-i\omega\tau_x} + P_2(i\omega)e^{-i\omega\tau_y} + P_3(i\omega)e^{-i\omega(\tau_x+\tau_y)} = 0$$

where

$$P_0(i\omega) = i\omega - \beta_x\beta_y,\ P_1(i\omega) = i\alpha_x\omega,\ P_2(i\omega) = i\alpha_y\omega,\ P_3(i\omega) = \alpha_x\alpha_y$$

- The characteristic equation can be rewritten as

$$P_0(i\omega) + P_1(i\omega)e^{-i\omega\tau_x} + \left[P_2(i\omega) + P_3(i\omega)e^{-i\omega\tau_x}\right] e^{-i\omega\tau_y} = 0$$

Two delay Model

- Since $\left| e^{-i\omega\tau_y} \right| = 1$,

$$\left| P_0(i\omega) + P_1(i\omega)e^{-i\omega\tau_x} \right| = \left| P_2(i\omega) + P_3(i\omega)e^{-i\omega\tau_x} \right|$$

- Squaring both sides generates the equivalent forms,

$$\begin{aligned} & \left(P_0(i\omega) + P_1(i\omega)e^{-i\omega\tau_x}\right)\left(\bar{P}_0(i\omega) + \bar{P}_1(i\omega)e^{i\omega\tau_x}\right) \\ = \quad & \left(P_2(i\omega) + P_3(i\omega)e^{-i\omega\tau_x}\right)\left(\bar{P}_2(i\omega) + \bar{P}_3(i\omega)e^{i\omega\tau_x}\right) \end{aligned}$$

where over-bar indicates complex conjugate.

Two delay Model

- the characteristic equation can be reduced to

$$|P_0|^2 + |P_1|^2 - |P_2|^2 - |P_3|^2 = 2A_x(\omega)\cos\omega\tau_x - 2B_x(\omega)\sin\omega\tau_x$$

where

$$A_x(\omega) = \mathrm{Re}\left[P_2\bar{P}_3 - P_0\bar{P}_1\right] = 0$$

and

$$B_x(\omega) = \mathrm{Im}\left[P_2\bar{P}_3 - P_0\bar{P}_1\right] = \alpha_x\omega\left(\alpha_y^2 - \beta_x\beta_y - \omega^2\right)$$

- The characteristic equation is now

$$|P_0|^2 + |P_1|^2 - |P_2|^2 - |P_3|^2 = -2B_x(\omega)\sin\omega\tau_x$$

- We consider (1) $B_x(\omega) = 0$ and (2) $B_x(\omega) \neq 0$.

$B_x(\omega) = 0$

- Assumption $\alpha_x = \alpha_y = \alpha$ and $\beta_x = \beta_y = \beta$

$$B_x(\omega) = \alpha\omega\left(\alpha^2 - \beta^2 - \omega^2\right) = 0$$

leads to

$$\omega^* = \sqrt{\alpha^2 - \beta^2} > 0$$

- the characteristic equation

$$P_0(i\omega) + P_1(i\omega)e^{-i\omega\tau_x} + \left[P_2(i\omega) + P_3(i\omega)e^{-i\omega\tau_x}\right]e^{-i\omega\tau_y} = 0$$

can be reduced to

$$e^{-i\omega\tau_y} = -\frac{P_0(i\omega) + P_1(i\omega)e^{-i\omega\tau_x}}{P_2(i\omega) + P_3(i\omega)e^{-i\omega\tau_x}}$$

Two delay Model:

- By Euler's formula, the last equation can be written as

$$\cos \omega\tau_y - i \sin \omega\tau_y = -\frac{m_x}{d_x} - i\frac{n_x}{d_x}$$

where

$$d_x = \alpha^2 \left[(\cos \omega\tau_x)^2 + (\omega - \alpha \sin \omega\tau_x)^2 \right] > 0$$

$$m_x = -(\alpha\beta)^2 \cos \omega\tau_x$$

$$n_x = \alpha \left[2\alpha^2\omega - \alpha \left(2\alpha^2 - \beta^2 \right) \sin \omega\tau_x \right]$$

- Comparing both sides, we have

$$\cos \omega\tau_y = -\frac{m_x}{d_x} \text{ and } \sin \omega\tau_y = \frac{n_x}{d_x}$$

Two delay Model:

- Parameter Specification (Shibata-Saito (1980))

$$a_{11} = a_{22} = 2, \; a_{12} = a_{21} = 1 \text{ and } \varepsilon_1 = \varepsilon_2 = 1$$

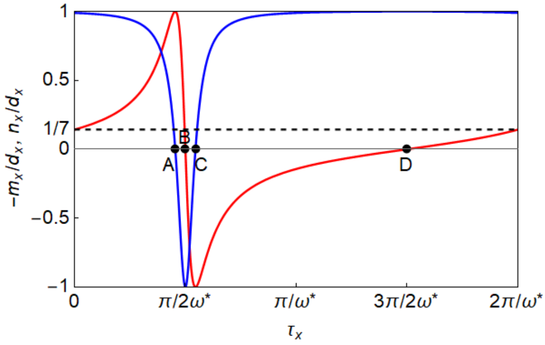
imply

$$x^e = \frac{1}{3} \text{ and } y^e = \frac{1}{3}$$

$$\alpha_x = \alpha_y = \alpha = \frac{4}{3} \text{ and } \beta_x = \beta_y = \beta = \frac{2}{3}$$

Two delay Model:

$$-\frac{m_x}{d_x}\text{: red curve, } \frac{n_x}{d_x}\text{: blue curve}$$



- In $[0, \tau_x^A]$, it is seen that

$$\cos \omega \tau_y = -\frac{m_x}{d_x} > 0 \text{ and } \sin \omega \tau_y = \frac{n_x}{d_x} > 0$$

- Both inequalities implies

$$\frac{\pi}{2} < \omega \tau_y < \pi$$

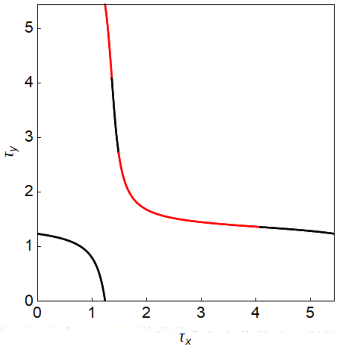
- Solving these equations for τ_y gives

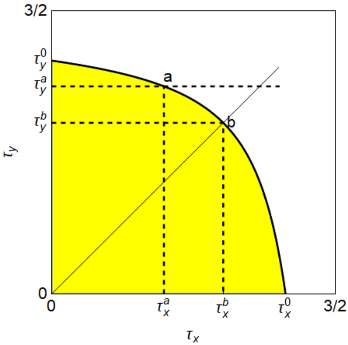
$$\tau_y^C(\tau_x) = \frac{1}{\omega^*} \cos^{-1}\left(-\frac{m_x}{d_x}\right) \text{ and } \tau_y^S(\tau_x) = \frac{1}{\omega^*} \sin^{-1}\left(-\frac{n_x}{d_x}\right)$$

- In $[\tau_x^A, \tau_x^B]$, $[\tau_x^B, \tau_x^C]$, $[\tau_x^C, \tau_x^D]$, $[\tau_x^D, 2\pi/\omega^*]$, the same procedure is repeated to obtaine the corresponding values of τ_y against the values of τ_x

Two delay Model:

- Stability switching curve





Two delay Model:

- $|B_x(\omega)| > 0$: There is no (τ_x, τ_y) solving the characteristic equation,

$$|P_0|^2 + |P_1|^2 - |P_2|^2 - |P_3|^2 = -2B_x(\omega)\sin\omega\tau_x$$

- From the form of the characteristic equation

$$P_0(i\omega) + P_1(i\omega)e^{-i\omega\tau_x} + P_2(i\omega)e^{-i\omega\tau_y} + P_3(i\omega)e^{-i\omega(\tau_x+\tau_y)} = 0$$

we can have

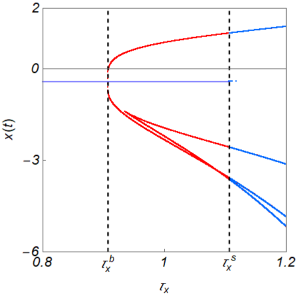
$$P_0 + P_2 e^{-i\omega\tau_y} + \left[P_1 + P_3 e^{-i\omega\tau_y}\right] e^{-i\omega\tau_x} = 0$$

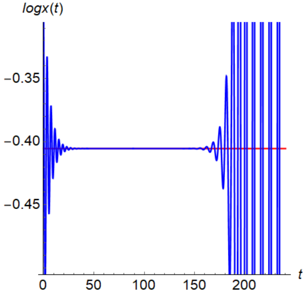
that generates the following form of the characteristic equation,

$$|P_0|^2 - |P_1|^2 + |P_2|^2 - |P_3|^2 = -2B_y(\omega)\sin\omega\tau_y$$

Simulation 1: Symmetric Case

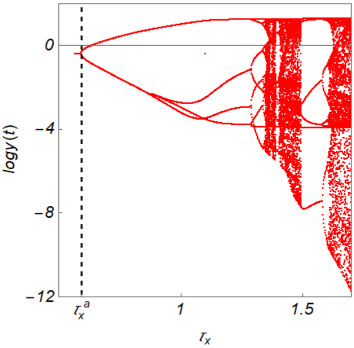
- Parameter Specificaiton I: $a_{11} = a_{22} = 2,\ a_{12} = a_{21} = 1,\ \varepsilon_1 = \varepsilon_2 = 2$
- red curve different initial functions of $\phi_x = x^e + 0.2$ and $\phi_y = y^e + 0.1$
- Blue curve same initial functions of $\phi_x = x^e + 0.1$ and $\phi_y = y^e + 0.1$

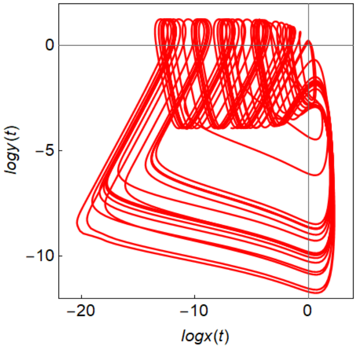




Simulation 2: Non-Symmetric Case

- Bifurcation diagram: $\tau_y = 1.1$(fixed), τ_x increases from 0.8 to 1.7
- Phase diagram; $\tau_x = 1.15$ and $\tau_y = 1.1$





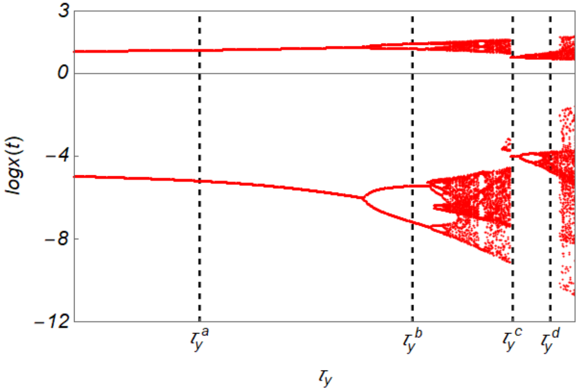
Simulation 3: Reproductions of Shibata-Saito's results

- Shibata-Saito (1980) obtain the following results:

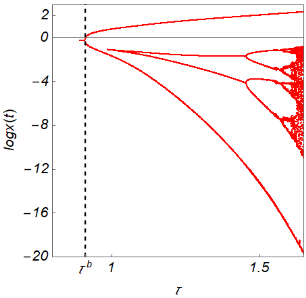
$$
\begin{aligned}
(\tau_x, \tau_y) &= (1.6, 0.6) &&\Longrightarrow \text{a lilmit cycle} \\
(\tau_x, \tau_y) &= (1.6, 0.77) &&\Longrightarrow \text{a two-period cycle} \\
(\tau_x, \tau_y) &= (1.6, 0.85) &&\Longrightarrow \text{complicated dynamics} \\
(\tau_x, \tau_y) &= (1.6, 0.87) &&\Longrightarrow \text{complicated dynamics}
\end{aligned}
$$

Simulation 3: Reproductions of Shibata-Saito's results

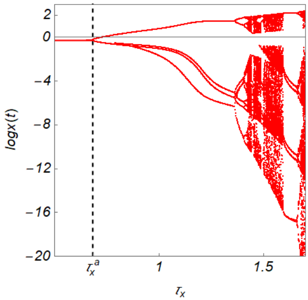
- Bifurcation diagram with respect to τ_y with $\tau_x = 1.6$



- Parameter Specification II:
  $a_{11} = 2,\ a_{12} = 5/2,\ a_{21} = a_{22} = 1,\ \varepsilon_1 = \varepsilon_2 = 2$



$\tau^h \simeq 0.911,\ \tau_x = \tau_y$



$\tau_y^a = 1.1,\ \tau_x \neq \tau_y^a$

Concluding Remarks

(1) No delay model: the steady state is locally asymptotically stable.

(2) One delay model: stability loss and the birth of a hopf bifurcation.

(3) Two delay model: a stability switching curve is analytically obtained.