The stochastic sensitivity of bull- and bear states in an asset market

Jochen Jungeilges [1,2] Tatyana Ryazanova [2]

[1] University of Agder, School of Business and Law, Kristiansand S, Norway
[2] Ural Federal University, Institute of Natural Sciences and Mathematics, Ekaterinburg, Russia

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Background

Research area: Asset markets with heterogenous investors
Seminal model: Day and Huang (1990), Huang and Day (1993).

- Tramontana et al. (2010)
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- Sushko et al. (2015)
- Panchuk et al. (2018)

Quest for future efforts:

1. Allow for asymmetric response around the fundamental value.
2. Diversify no-trade intervals.
3. Intensify the stochastic modelling effort.

Scope of the project

Model: A stochastic DH asset-price process

- Asymmetries in trading behavior within agent-type
- No-trade intervals of agent-types do not coincide
- Types of noise: additive and parametric

Goal: Further our understanding of the asset price dynamics in speculative markets

1. Study the dynamics of the deterministic map (5 linear pieces map with 2 discontinuities).
2. Analyze the sensitivity of stochastic equilibria.
3. Identify different types of transitions.
4. Unravel the "genesis" of the transitions.

Method: Indirect method, stochastic sensitivity function (SSF) Milstein and Ryashko (1995)

Fundamentalists

Definition 1

The excess demand of α-investors is given by

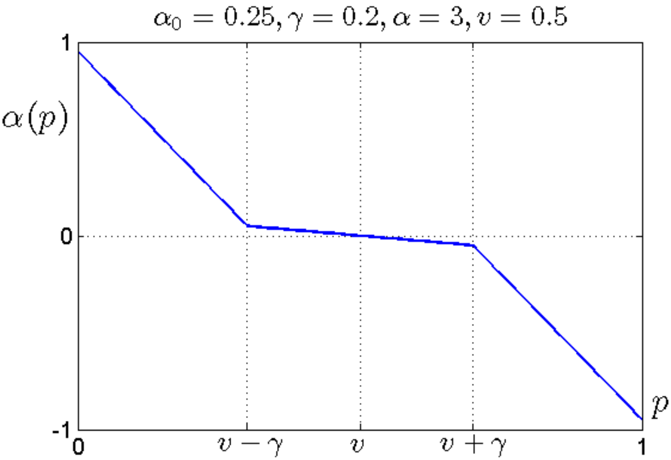
$$\alpha(p) = \begin{cases} \alpha_0\gamma - \alpha_l(p - v + \gamma), & p \leq v - \gamma; \\ \alpha_0(v - p), & v - \gamma < p < v + \gamma; \\ -\alpha_0\gamma - \alpha_u(p - v - \gamma), & p \geq v + \gamma. \end{cases}$$

$v \in (0, 1)$, $0 < \gamma < min(v, 1 - v)$, $\alpha_0 \geq 0$, $\alpha_l \geq 0$, $\alpha_u \geq 0$.

Assumption 1

$\alpha_l \geq \alpha_0$, $\alpha_u \geq \alpha_0$

Fundamentalists



Chartists

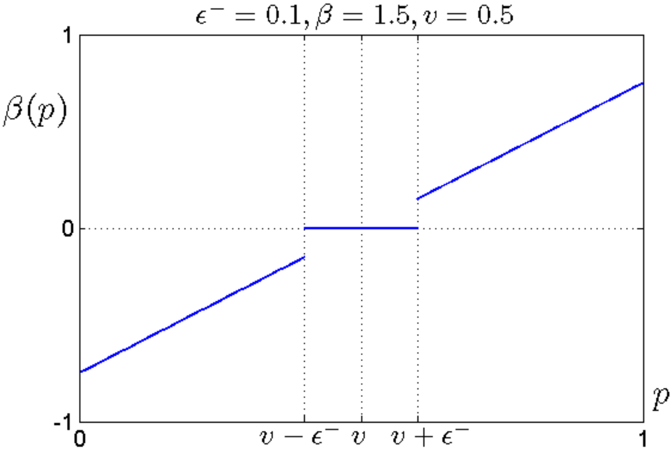
Definition 2

The excess demand of the β-investors is given by

$$\beta(p) = \begin{cases} \beta_l\,(p-v), & p \leq v-\epsilon^-; \\ \beta_u\,(p-v), & p \geq v+\epsilon^-; \\ 0, & \text{o.w.} \end{cases}$$

with $p \in (0,1)$, $0 < \epsilon^- < min(v, 1-v)$, $\beta_l \geq 0$, $\beta_u \geq 0$.

Chartists



Deterministic Price process

Price process

Relying on Definitions 1 and 2 the asset price process can be given as

$$p_{t+1} = f(p_t) = p_t + \alpha(p_t; \gamma, \alpha_0, \alpha_l, \alpha_u, v) + \beta(p_t, \epsilon^-; \beta_l, \beta_u, v) \quad (1)$$

with $p_0 \in (0,1)$.

Case: $\gamma > \epsilon^-$

If $\gamma > \epsilon^-$ then the price process is given by $p_{t+1} = f(p_t)$ with

$$f(p) = \begin{cases} f_1(p) = (1 - \alpha_l + \beta_l)p + \alpha_0\gamma + \alpha_l v - \alpha_l\gamma - \beta_l v, & 0 \le p < v - \gamma; \\ f_2(p) = (1 - \alpha_0 + \beta_l)p + \alpha_0 v - \beta_l v, & v - \gamma \le p \le v - \epsilon^-; \\ f_3(p) = (1 - \alpha_0)p + \alpha_0 v, & v - \epsilon^- < p < v + \epsilon^-; \\ f_4(p) = (1 - \alpha_0 + \beta_r)p + \alpha_0 v - \beta_r v, & v + \epsilon^- \le p \le v + \gamma; \\ f_5(p) = (1 - \alpha_r + \beta_r)p - \alpha_0\gamma + \alpha_r v + \alpha_r\gamma - \beta_r v, & v + \gamma < p \le 1. \end{cases}$$

$\gamma > \epsilon^-$: Equilibria

Assumption 2

$\alpha > \beta + 1$

Assumption 3

$\alpha_r = \alpha_l = \alpha$ and $\beta_r = \beta_l = \beta$.

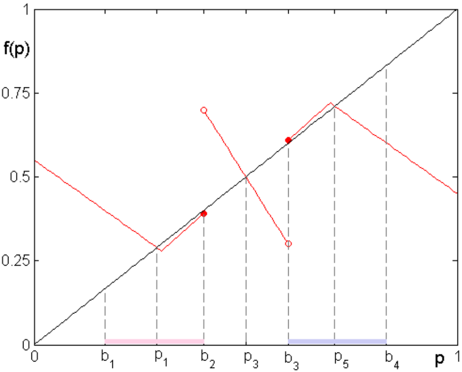
Result 1

Let $\underline{\delta} = \max\left(\alpha - \frac{v}{\gamma}(\alpha - \beta), \alpha - \frac{1-v}{\gamma}(\alpha - \beta)\right)$. Suppose Assumptions 2 and 3 hold. If $\gamma < v$ and $\alpha_0 \in (\underline{\delta}, \beta)$ then the equilibria

$$p_1 = v - \frac{\gamma(\alpha - \alpha_0)}{\alpha - \beta}, \; p_3 = v, \; p_5 = v + \frac{\gamma(\alpha - \alpha_0)}{\alpha - \beta}$$

exist. The equilibria p_1 and p_5 are locally stable if $\beta > \alpha - 2$ and p_3 is stable if $0 < \alpha_0 < 2$.

Immediate basins



$$b_1 = v - \frac{\epsilon^- - (\alpha - \alpha_0)\gamma}{1 - \alpha + \beta} \quad b_2 = v - \epsilon^- \quad b_3 = v + \epsilon^- \quad b_4 = v + \frac{\epsilon^- - (\alpha - \alpha_0)\gamma}{1 - \alpha + \beta}$$

The stochastic model

Stochastic price process

Relying on Definitions 4 and 6 the asset price process can be given as

$$p_{t+1} = p_t + \alpha(p_t; \gamma, \alpha_0, \alpha, v) + \beta(p_t, \epsilon^-; \beta + \pi\xi_t, v) + \varepsilon\xi_t \tag{2}$$

with $p_0 \in (0,1)$, $\varepsilon, \pi \geq 0$, $\xi_t \sim N(0,1)$.

$\pi = 0, \varepsilon > 0$	additive shocks
$\pi > 0, \varepsilon = 0$	parametric shocks
$\pi > 0, \varepsilon > 0$	mixture

Sensitivity analysis via SSF

We can represent (2) as

$$p_{t+1} = f(p_t) + \varepsilon g(p_t)\xi_t \tag{3}$$

where $g(\bullet)$ denotes a smooth function.

Assumption 4

For $\varepsilon = 0$ (3) has an exponentially stable equilibrium $\bar{p}$.

Let $p_t(\varepsilon)$ be the solution of (3) with $p_0(\varepsilon) = \bar{p} + \varepsilon\nu_0$ then

$$z_t = \lim_{\varepsilon \to 0} \frac{p_t(\varepsilon) - \bar{p}}{\varepsilon}$$

characterizes the sensitivity of the price equilibrium to i.i.d. shocks.

$$z_{t+1} = f'(\bar{p})z_t + g(\bar{p})\xi_t$$

Sensitivity continued

Focus on the dynamics of second moments: $V_t = \mathbb{E}[z_t^2]$

$$V_{t+1} = [f'(\bar{p})]^2 V_t + g(\bar{p})$$

Assumption 4 $\Rightarrow \ \mid f'(\bar{p}) \mid < 1$

$$\omega = \lim_{t\to\infty} V_t = \frac{g^2(\bar{p})}{1 - [f'(\bar{p})]^2}$$

Confidence interval: $\bar{p} \pm k\varepsilon\sqrt{2\omega}$ where $k = erf^{-1}(0.99)$

Remarks:

- ω and ε define the borders of the confidence interval for $\bar{p}$.
- $D = \varepsilon^2\omega$ is related to the variance matrix of the stationary density.
- ω is the stochastic sensitivity function (SSF) for the attractor $\bar{p}$.
- The SSF relates the intensity of stochastic signal ε^2.

Sensitivity analysis for the stochastic price process

Case $\gamma > \epsilon^-$: additive noise, i.e. $g(\bar{p}) = 1$

$$f'(p) = \begin{cases} 1 - \alpha + \beta, & 0 \leq p < v - \gamma; \\ 1 - \alpha_0 + \beta, & v - \gamma \leq p \leq v - \epsilon^-; \\ 1 - \alpha_0, & v - \epsilon^- < p < v + \epsilon^-; \\ 1 - \alpha_0 + \beta, & v + \epsilon^- \leq p \leq v + \gamma; \\ 1 - \alpha + \beta, & v + \gamma < p \leq 1. \end{cases}$$

$$\omega_1 = \frac{1}{1 - (1 - \alpha + \beta)^2} \quad \omega_3 = \frac{1}{1 - (1 - \alpha_0)^2} \quad \omega_5 = \frac{1}{1 - (1 - \alpha + \beta)^2}$$

Graphs of SSFs

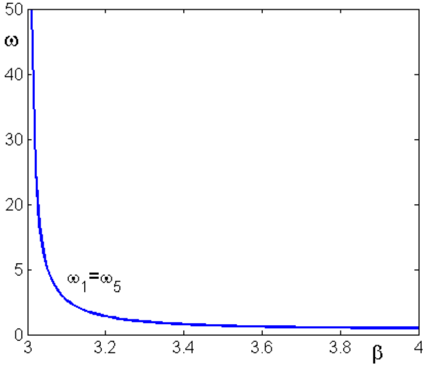


Figure: ω_1 for $\alpha = 5$ and $\alpha - 2 < \beta < \alpha - 1$

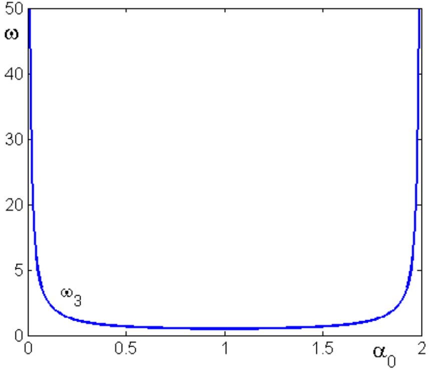


Figure : ω_3 for $0 < \alpha_0 < 2$

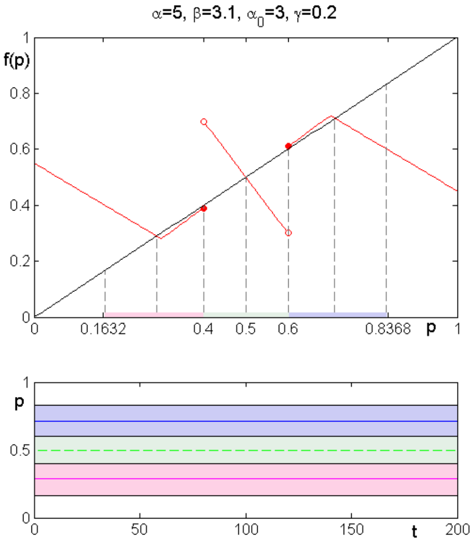
Confidence intervals

$$p_1 : \quad v - \frac{\gamma(\alpha - \alpha_0)}{\alpha - \beta} \pm k\varepsilon\sqrt{\frac{2}{1 - (1 - \alpha + \beta)^2}}$$

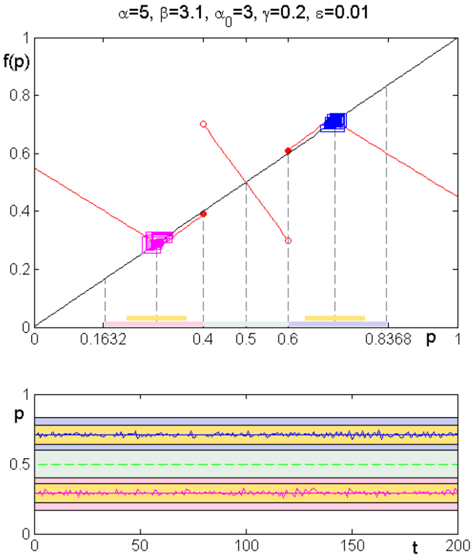
$$p_3 : \quad v \pm k\varepsilon\sqrt{\frac{2}{1 - (1 - \alpha_0)^2}}$$

$$p_5 : \quad v + \frac{\gamma(\alpha - \alpha_0)}{\alpha - \beta} \pm k\varepsilon\sqrt{\frac{2}{1 - (1 - \alpha + \beta)^2}}$$

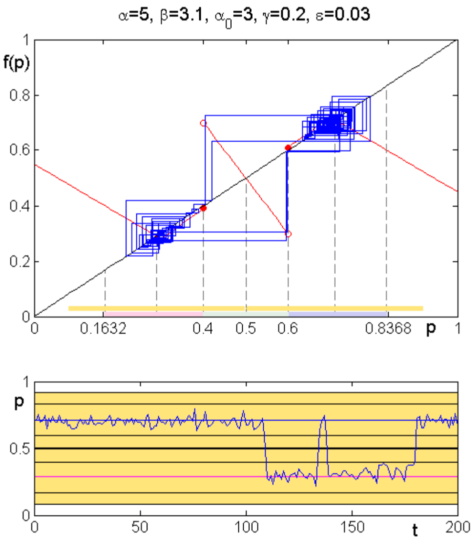
Case 2: stable: p_1, p_5 unstable p_3; $\varepsilon = 0$



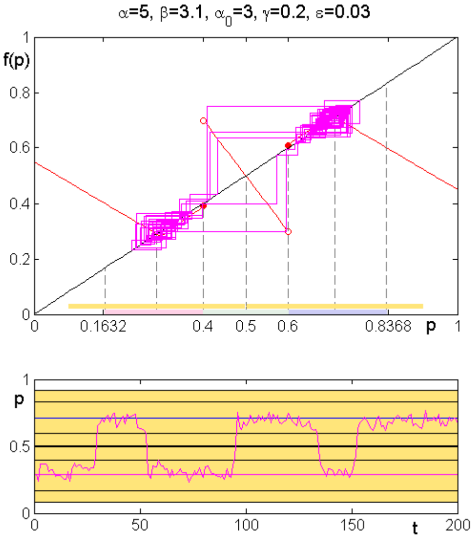
Case 2: stable: p_1, p_5 unstable p_3; $p_0 = p_1(p_5)$, $\varepsilon = 0.01$



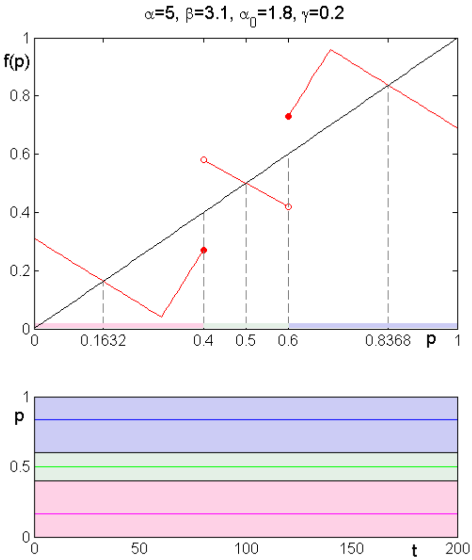
Case 2: stable: p_1, p_5 unstable p_3; $p_0 = p_5$, $\varepsilon = 0.03$



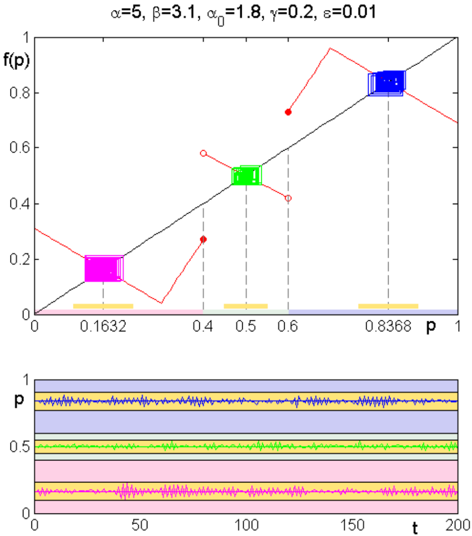
Case 2: stable: p_1, p_5 unstable p_3; $p_0 = p_1$, $\varepsilon = 0.03$



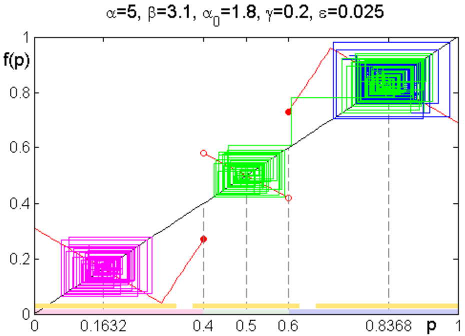
Case 3: stable p_1, p_3, p_5, $\varepsilon = 0$

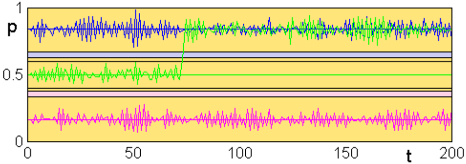


Case 3: stable p_1, p_3, p_5, $p_0 = p_i$, $\varepsilon = 0.01$

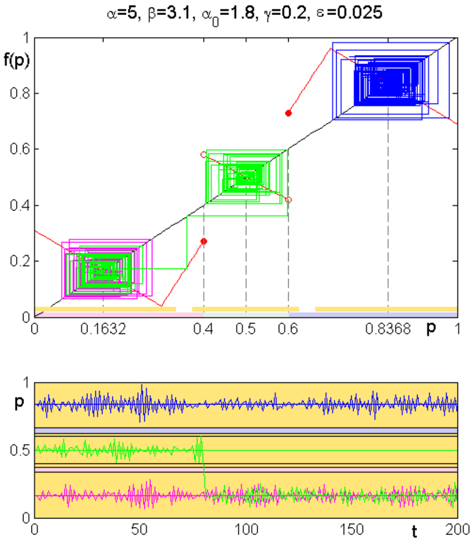


Case 3: stable p_1, p_3, p_5, $p_0 = p_i$, $\varepsilon = 0.025$





Case 3: stable p_1, p_3, p_5, $p_0 = p_i$, $i \in \{1, 3, 5\}$, $\varepsilon = 0.025$



Conclusion

- Key elements in the genesis of transition between boom and bust states:
  1. immediate basins of attraction,
  2. confidence ellipses of attractors
- The noise levels at which transitions become likely depends on trading intensities.
- In the case of additive noise, transitions between equilibria might occur even if all equilibria are stable.
- Transitions are more likely to occur under parametric noise than under additive noise (work in progress).

References I

Day, R. H. and Huang, W. (1990). Bulls, bears and market sheep. *Journal of Economic Behavior & Organization*, 14(3):299–329.

Huang, W. and Day, R. (1993). Chaotically switching bear and bull markets: The derivation of stock price distributions from behavior rules. In Day, R. and Chen, P., editors, *Nonlinear Dynamics and Evolutionary Economics*. Oxford University Press.

Milstein, G. and Ryashko, L. (1995). The first approximation in the quasipotential problem of stability of non-degenerate systems with random perturbations. *Applied Mathematics and Mechanics*, 1(59):51–XX. in Russian.

Panchuk, A., Sushko, I., and Westerhoff, F. (2018). A financial market model with two discontinuities: Bifurcation structures in the chaotic domain. *Chaos*, 28(055908).

References II

Sushko, I., Tramontana, F., Westerhoff, F., and Avrutin, V. (2015). Symmetry breaking in a bull and bear financial market model. *Chaos, Solitons & Fractals*, 79:57 – 72. Proceedings of the MDEF (Modelli Dinamici in Economia e Finanza Dynamic Models in Economics and Finance) Workshop, Urbino 18th20th September 2014.

Tramontana, F., Gardini, L., and Westerhoff, F. (2011). Heterogeneous speculators and asset price dynamics: Further results from a one-dimensional discontinuous piecewise-linear map. *Computational Economics*, 38(3):329.

Tramontana, F., Westerhoff, F., and Gardini, L. (2010). On the complicated price dynamics of a simple one-dimensional discontinuous financial market model with heterogeneous interacting traders. *Journal of Economic Behavior & Organization*, 74(3):187 – 205.

References III

Tramontana, F., Westerhoff, F., and Gardini, L. (2013). The bull and bear market model of huang and day: Some extensions and new results. *Journal of Economic Dynamics and Control*, 37(11):2351 – 2370.

Tramontana, F., Westerhoff, F., and Gardini, L. (2014). One-dimensional maps with two discontinuity points and three linear branches: Mathematical lessons for understanding the dynamics of financial markets. *Decisions in Economics and Finance*, 37.

Tramontana, F., Westerhoff, F., and Gardini, L. (2015). A simple financial market model with chartists and fundamentalists: Market entry levels and discontinuities. *Mathematics and Computers in Simulation*, 108:16 – 40. Complex Dynamics in Economics and Finance - Papers of the MDEF 2012 Workshop.