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# Memory Effects on Binary Choices with Impulsive Agents: Bistability and a new BCB structure

**Laura Gardini**

*Dept of Economics, Society and Politics, University of Urbino, Italy*

**Arianna Dal Forno**

*Department of Economics, University of Molise, Italy*

**Ugo Merlone**

*Department of Psychology, University of Torino, Italy,*

# Content of the talk

- After the works by Shelling (1973), several authors have considered models representing impulsive choices by different kinds of groups. Following the ideas and the model proposed in (Bischi et al. 2009) represented by a one-dimensional discontinuous piecewise linear map, memory has been introduced linking the next output to the present and the last state. This results in a two-dimensional discontinuous piecewise linear map, whose dynamics and bifurcations are investigated.
- 1D map, 2D extension and motivation
- Existence and stability of cycles
- Periodicity regions, organized as in the period adding bifurcation structure with additional new elements
- Differently from the one-dimensional case, coexistence of two attracting cycles is now possible in many regions of the parameter space
- Structure of the basins of attraction when two attractors coexist

# description of the 1D map

## the 1D PWL system

In (Bischi et al. 2009) we have considered a population with a unitary continuum of players in $[0,1]$ and agents choose strategies from a set of two actions $A$ or $B$ as a result of a binary choices process; $x_t \in [0,1]$ denotes the fraction of agents playing strategy $A$ while $(1-x_t)$ the proportion of those playing strategy $B$ at the same time. Individual payoff are assumed linear:

$$U_A(x_t) = A(x_t) = p_A x_t + q_A \quad , \quad U_B(x_t) = B(x_t) = p_B x_t + q_B \ .$$

We assume that agents are homogeneous and myopic. If a fraction $x_t$ of players are playing strategy $A$ and $U_A(x_t) > U_B(x_t)$ then a fraction of the $(1-x_t)$ agents that are playing strategy $B$ will switch to $A$ in the following turn. Similarly, if $U_A(x_t) < U_B(x_t)$, then a fraction of the $x_t$ players that are playing $A$ will switch to strategy $B$. In other words, at any time $t$ agents decide their action for the period $t+1$ comparing $A(x_t)$ and $B(x_t)$ according to a map $x_{t+1} = T(x_t)$

# description of the 1D map

## the 1D PWL system

$x_{t+1} = T(x_t)$ is assumed as follows:

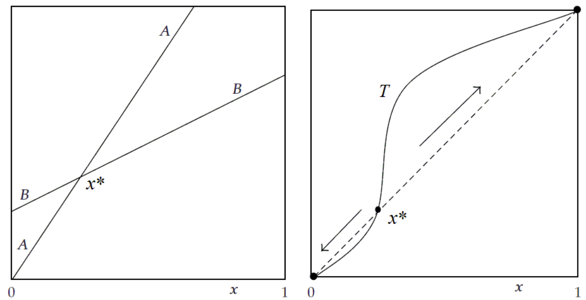
$$T(x_t) = \begin{cases} x_t - \delta_B g[\lambda(B(x_t) - A(x_t))]x_t & \text{if } U_B(x_t) > U_A(x_t) \\ x_t + \delta_A g[(\lambda(A(x_t) - B(x_t))](1 - x_t) & \text{if } U_B(x_t) < U_A(x_t) \end{cases}$$

where $\delta_B$ and $\delta_A$ represent the agents' propensity to switch to the other strategy, $\delta_A, \delta_B \in [0, 1]$; $g : R \rightarrow [0, 1]$ is a continuous and increasing function such that $g(0) = 0$ and $\lim_{z\rightarrow\infty} g(z) = 1$ which modulates how the fraction of switching agents depends on the difference between the payoffs. The function $g(z) = \frac{2}{\pi} arctan(z)$ is a prototype.
The parameter $\lambda$ represents the switching intensity or speed of reaction. Larger values of $\lambda$ can be interpreted in terms of impulsivity. According to the Clinical Psychology literature (Patton et al. 1995) impulsivity leads agents to act on the spur of the moment and lack of planning.

## the 1D PWL system

The NE of the game is $x^* \in (0,1)$ such that $A(x^*) = B(x^*)$ which leads, assuming $p_A \neq p_B$, to the point $x^* = -\frac{q_B - q_A}{p_B - p_A} = -\frac{\Delta_q}{\Delta_p}$

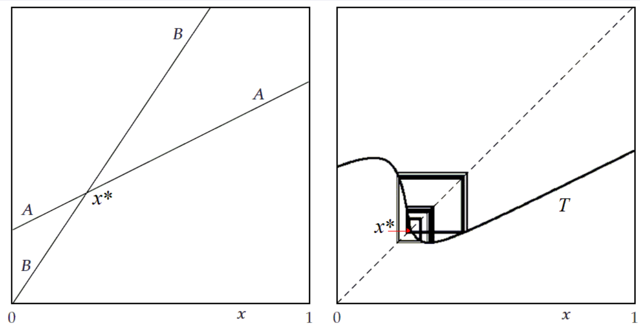
The example proposed by Schellig with $\Delta_p < 0$ and $\Delta_q > 0$ leads to payoff functions and 1D map $T$ of this kind:




where $A(x) = 1.5x$ and $B(x) = 0.25 + 0.5x$, $\delta_A = 0.3$, $\delta_B = 0.7$ and $\lambda = 20$

## the 1D PWL system

Differently, the case with $\Delta_p > 0$ and $\Delta_q < 0$ leads to payoff functions and 1D map of this kind:

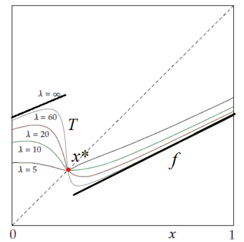



where $B(x) = 1.5x$ and $A(x) = 0.25 + 0.5x$, $\delta_A = 0.5 = \delta_B$ and $\lambda = 35$.

# description of the 1D map

## the 1D PWL system

The effect of the parameter $\lambda$ is to steepen the function $T$ in a neighborhood of $x^*$ :



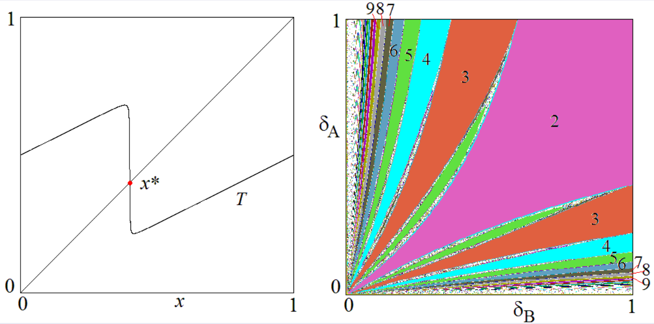

and the dynamics for large values of $\lambda$ can be well approximated by the piecewise linear map with a discontinuity in $x^*$, representing the case of impulsive choices:

$$x_{t+1} = f(x_t) = \begin{cases} x_t - \delta_B x_t & \text{if} \quad x_t > x^* \\ x_t + \delta_A (1 - x_t) & \text{if} \quad x_t < x^* \end{cases}$$

# description of the 1D map

## the 1D PWL system

We can appreciate the similarity via a two-dim. bifurcation diagram:

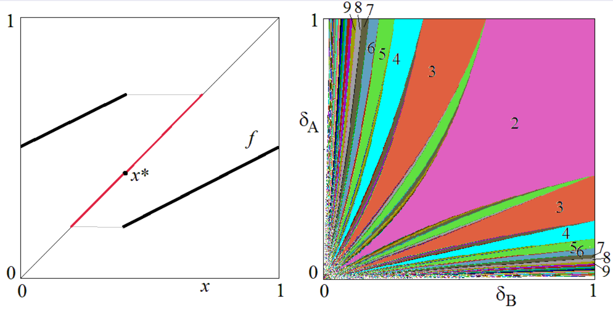



here $x_{t+1} = T(x_t)$ with $\lambda = 900$. The bifurcation curves in the parameter plane are related to fold bifurcations of the cycles and sequences of flip bifurcations. Several wide periodicity regions related to stable cycles exist.

# description of the 1D map

## the 1D PWL system

compared with the impulsive case, the discontinuous map $x_{t+1} = f(x_t)$:




the NE $x^*$ is no longer an equilibrium of the dynamic game, the bifurcations are now related to BCBs due to the collision of a periodic point of the cycles with the discontinuity point $x^*$ (Leonov 1960a,b, some families of Lorenz maps in Homburg 1996, Keener 198, Gardini et al. 2010, Avrutin et al. 2010, 2019)

# The model with memory

We are interested in extending the impulsive model in order to take into account some information. Although agents are impulsive and follow their utility almost directly, the knowledge not only of the present value but also a short memory as the previous state, may be taken into account, leading to impulsive behavior with a memory effect. In the existing literature the effects of memory seem not to be univocal, from the "irrelevance of memory" described in Cavagna (1999) to the "relevance of memory" as reported by Challet and Marsili (2000).
We can observe results which merge the two different kinds of interpretation. In fact, with a low weight given to the previous state the system evolves as in the absence of memory while increasing the weight given to the past state the role of memory comes to play.
The use of the limiting discontinuous map (representing the impulsive behavior) in place of the smooth one has the advantage to keep the system simpler to analyze, although the discontinuity leads to a class of maps still not well studied, and indeed we shall observe new bifurcation phenomena, which are worth to be investigated in more detail.

# The model with memory

## the 2D PWL system

we keep $A(x_t) = p_A x_t + q_A$ and $B(x_t) = p_B x_t + q_B$ but the utility function governing the agents' behavior is modeled as the weighted average of the current payoff and the one previously observed:

$$U_A(x_t, x_{t-1}) = (1-\omega)A(x_t) + \omega A(x_{t-1})$$
$$U_B(x_t, x_{t-1}) = (1-\omega)B(x_t) + \omega B(x_{t-1})$$

Then agents update their choice at any time $t$ via $x_{t+1} = f(x_t, x_{t-1})$:

$$f(x_t, x_{t-1}) = \begin{cases} f_B(x_t) = (1-\delta_B)x_t & \text{if } U_B(x_t, x_{t-1}) > U_A(x_t, x_{t-1}) \\ f_A(x_t) = (1-\delta_A)x_t + \delta_A & \text{if } U_B(x_t, x_{t-1}) < U_A(x_t, x_{t-1}) \end{cases}$$

The two functions $f_B(x_t)$ and $f_A(x_t)$ depend only on the state $x_t$ while the condition to get one or the other depends on the present state $x_t$ and the previous one $x_{t-1}$.

# The model with memory

## the 2D PWL system

For $\omega > 0$, let us introduce the variable $y_t = x_{t-1}$ then we can write our system as a two dimensional map $(x_{t+1}, y_{t+1}) = F(x_t, y_t)$ defined as follows

$$F : \begin{cases} x_{t+1} = \begin{cases} f_B(x_t) = (1-\delta_B)x_t & \text{if } (1-\omega)\Delta_p x_t + \omega\Delta_p y_t + \Delta_q > 0 \\ f_A(x_t) = (1-\delta_A)x_t + \delta_A & \text{if } (1-\omega)\Delta_p x_t + \omega\Delta_p y_t + \Delta_q < 0 \end{cases} \\ y_{t+1} = x_t \end{cases}$$

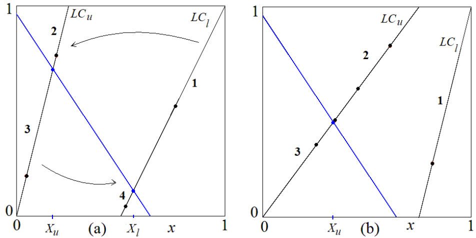
since $U_B(x_t, x_{t-1}) - U_A(x_t, x_{t-1}) = (1-\omega)\Delta_p x_t + \omega\Delta_p x_{t-1} + \Delta_q$, and we assume $\Delta_p = (p_B - p_A) > 0$ , $\Delta_q = (q_B - q_A) < 0$

# The 2D PWL system

## A line of discontinuity in the plane

we have a discontinuous PWL map $(x', y') = F(x, y)$ in the plane $(x, y)$ separated by a straight line with negative slope $y = -\frac{1-\omega}{\omega}x - \frac{\Delta_q}{\omega\Delta_p} = sx + \mu$

$$F(x,y) = \begin{cases} F_u(x,y) : \begin{cases} x' = (1-\delta_B)x \\ y' = x \end{cases} & \text{if } y > sx + \mu \\ F_l(x,y) : \begin{cases} x' = (1-\delta_A)x + \delta_A \\ y' = x \end{cases} & \text{if } y < sx + \mu \end{cases}$$



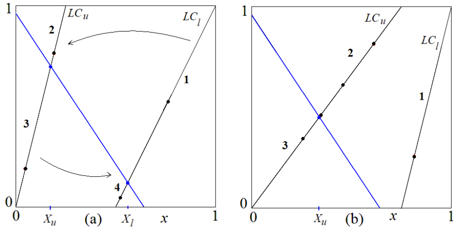
# The 2D PWL system

## Properties of map F

**1)** All the cycles belong to the lines $LC_u$ and $LC_l$ and have the symbolic sequences made up of blocks of type $\mathbf{12^n34^m}$, $n \geq 0$, $m \geq 0$ (For example, a cycle with symbolic sequence $\mathbf{12^2312^33}$ consists of the concatenation of $\mathbf{12^23}$ and $\mathbf{12^33}$**).** This also clarifies the possible border collision bifurcations: a periodic point with symbol 1 (4) can merge with $X_l$ from above (below), a periodic point with symbol 2 (3) can merge with $X_u$ from above (below).

**2)** At most two coexisting attracting cycles can exist, and no repelling cycle.

**3)** Repeated applications $F_u^n(x, y)$ and $F_l^m(x, y)$ are written explicitly



## Cycles of map F and related BCBs

A 2-cycle of map $F$ has symbolic sequence 13, it exists only for $0 < \omega < 0.5$ and the existence region is bounded by the sets $C_{31}$ and $C_{13}$ of equations:

$$C_{31}: \quad (1-\delta_B) = \frac{\mu - \delta_A}{\mu(1-\delta_A) - s\delta_A}$$

$$C_{13}: \quad (1-\delta_B) = \frac{\mu(1-\delta_A) + \delta_A - \delta_A(1 - s(1-\delta_A))}{(1-\delta_A)[\mu(1-\delta_A) + \delta_A]}$$

# The 2D PWL system

## Cycles of map F and related BCBs

Two kinds of 3-cycles of map $F$ exist, with symbolic sequence **123** and **134**. The existence region of the 3-cycle of map $F$ with symbolic sequence **123** is bounded by the sets $C_{123}$, $C_{312}$ and $C_{231}$ of equations:

$$C_{123}: \quad (1-\delta_B)^2 = \frac{\mu + \delta_A s}{\mu(1-\delta_A) + \delta_A}$$

$$C_{312}: \quad \delta_A = \frac{\mu - \mu(1-\delta_B)^2}{(1-\delta_B)[1 - s(1-\delta_B)] - \mu(1-\delta_B)^2}$$

$$C_{231}: \quad \delta_A = \frac{\mu - \mu(1-\delta_B)^2}{1 - s(1-\delta_B) - \mu(1-\delta_B)^2}$$

# The 2D PWL system

## Cycles of map F and related BCBs

The existence region of the 3-cycle of map $F$ with symbolic sequence **134** is bounded by the sets $C_{134}$, $C_{341}$ and $C_{413}$ of equations:

$$C_{134}: \quad (1-\delta_B) = \frac{[\mu(1-\delta_A)+\delta_A] - [1-(1-\delta_A)^2][1-s(1-\delta_A)]}{(1-\delta_A)^2[\mu(1-\delta_A)+\delta_A]}$$

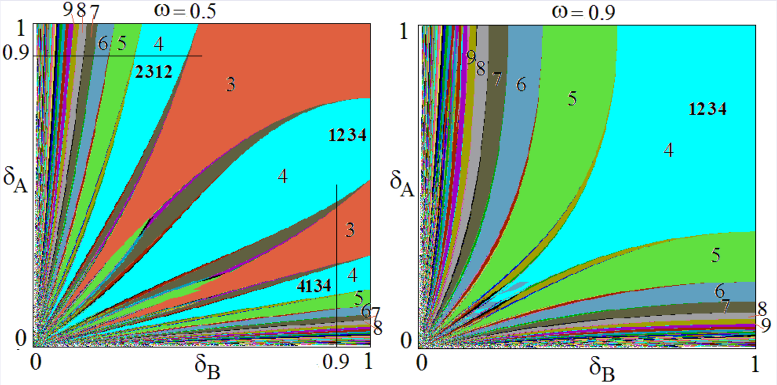
$$C_{341}: \quad (1-\delta_B) = \frac{\mu - 1 + (1-\delta_A)^2}{\mu(1-\delta_A)^2 - s[1-(1-\delta_A)^2]}$$

$$C_{413}: \quad (1-\delta_B) = \frac{\mu + s\delta_A}{\delta_A - s\delta_A(1-\delta_A) + (1-\delta_A)[\mu(1-\delta_A)+\delta_A]}$$

# The 2D PWL system

## Periodicity regions in a section of the parameter space

Three kinds of 4-cycles of map $F$ exist, with symbolic sequence $\mathbf{1234}$, $\mathbf{12^23}$, $\mathbf{134}^2$. While the 4-cycle $\mathbf{1234}$ has the existence region bounded by four sets, the other 4-cycles belong to two families of cycles having the symbolic sequence $\mathbf{12^n3}$ and $\mathbf{134^n}$ for $n \geq 2$, whose existence regions are bounded by three sets.



# The 2D PWL system

## Cycles of map F and related BCBs

The existence region of the 4-cycle of map $F$ with symbolic sequence **1234** is bounded by the four sets $C_{1234}$, $C_{4123}$, $C_{3412}$ *and* $C_{2341}$ of equations:

$$C_{1234}: \; (1-\delta_B)^2 = \frac{\mu(1-\delta_A)+\delta_A-[1-s(1-\delta_A)][2\delta_A-\delta_A^2]}{(1-\delta_A)^2[\mu(1-\delta_A)+\delta_A]}$$

$$C_{4123}: \; (1-\delta_B)^2 = \frac{\mu(1-\delta_A)+\delta_A-\delta_A[1-s(1-\delta_A)]}{\delta_A(1-\delta_A)[1-s(1-\delta_A)]+(1-\delta_A)^2[\mu(1-\delta_A)+\delta_A}$$

$$C_{3412}: \; [1-s(1-\delta_B)](1-\delta_B)[2\delta_A-\delta_A^2]+\mu(1-\delta_B)^2(1-\delta_A)^2-\mu=0$$

$$C_{2341}: \; (1-\delta_A)^2 = \frac{1-s(1-\delta_B)-\mu}{1-s(1-\delta_B)-\mu(1-\delta_B)^2}$$

## BCBs for a family of cycles

The existence regions of the cycles of map $F$ with symbolic sequence $\mathbf{312^n}$ for $n \geq 2$ are bounded by the sets $C_{312^n}$, $C_{2312^{n-1}}$ and $C_{12^n3}$ of equations:

$$C_{312^n}: \ \delta_A = \frac{\mu[1-(1-\delta_B)^{n+1}]}{(1-\delta_B)^n[1-s(1-\delta_B)-\mu(1-\delta_B)]}$$

$$C_{2312^{n-1}}: \ \delta_A = \frac{\mu[1-(1-\delta_B)^{n+1}]}{(1-\delta_B)^{n-1}[1-s(1-\delta_B)-\mu(1-\delta_B)]}$$

$$C_{12^n3}: \ (1-\delta_B)^{n+1} = \frac{\mu+s\delta_A}{\mu(1-\delta_A)+\delta_A}$$

# The 2D PWL system

## BCBs for a family of cycles

The existence regions of the cycle of map $F$ with symbolic sequence $\mathbf{134^n}$ for $n \geq 2$ are bounded by the sets $C_{134^n}$, $C_{4134^{n-1}}$ and $C_{34^n1}$ of equations:

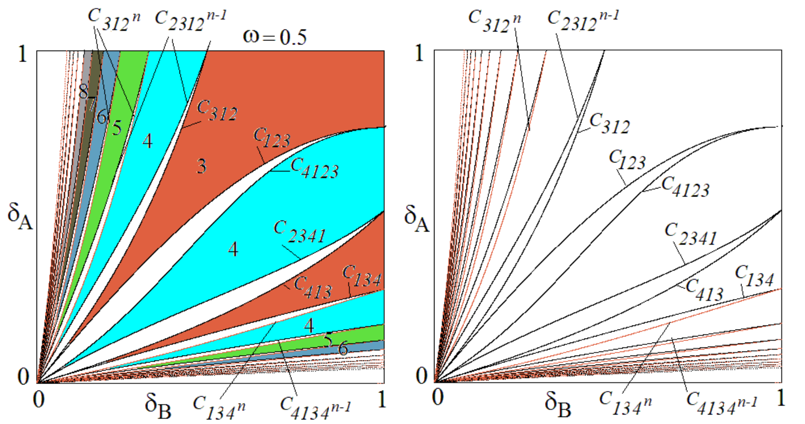
$$C_{134^n}: \ (1-\delta_B) = \frac{\mu(1-\delta_A)+\delta_A-[1-s(1-\delta_A)][1-(1-\delta_A)^{n+1}]}{(1-\delta_A)^{n+1}[\mu(1-\delta_A)+\delta_A]}$$

$$C_{4134^{n-1}}: \ (1-\delta_B) = \frac{\mu(1-\delta_A)+\delta_A-[1-s(1-\delta_A)][1-(1-\delta_A)^{n}]}{(1-\delta_A)^n[(\mu(1-\delta_A)+\delta_A)(1-\delta_A)+(1-s(1-\delta_A}$$

$$C_{34^n1}: \ (1-\delta_B) = \frac{\mu-1+(1-\delta_A)^{n+1}]}{(1-\delta_A)^{n+1}(\mu-s)}$$

## Cycles of map F and related BCBs

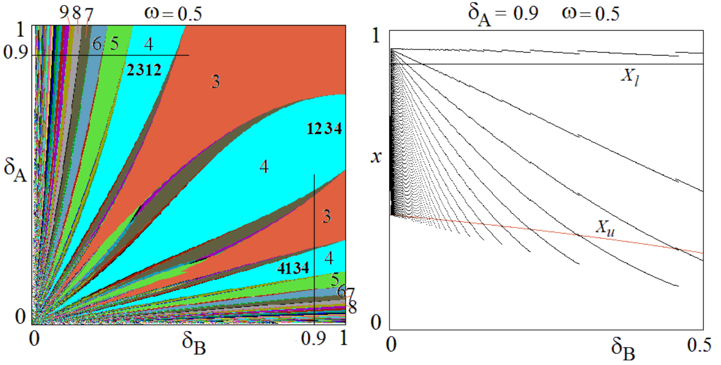
The two families of cycles having the symbolic sequence $\mathbf{12^n3}$ and $\mathbf{134^n}$ for $n \geq 2$, have the existence regions bounded by three sets (not visible in this section of the parameter space)



# The 2D PWL system

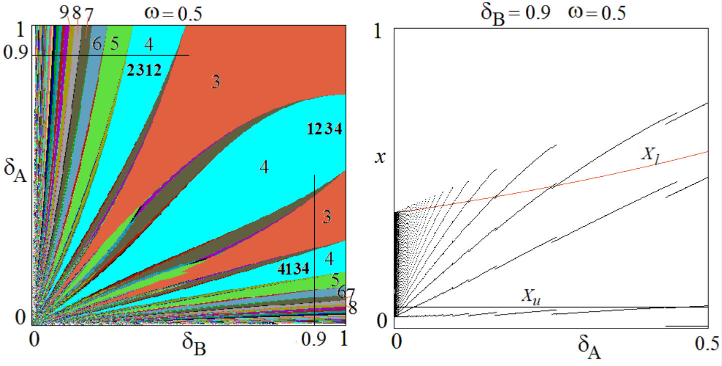
## Periodicity regions in a section of the parameter space

The period adding structure of the periodicity regions is clearly observable for $\delta_A < 1$. However, for $\delta_A = 1$ we have the period incrementing structure, only the principal regions exist, the regions are contiguous, the boundaries are those of the family symbolic sequence $\mathbf{312^n}$ for $n \geq 2$



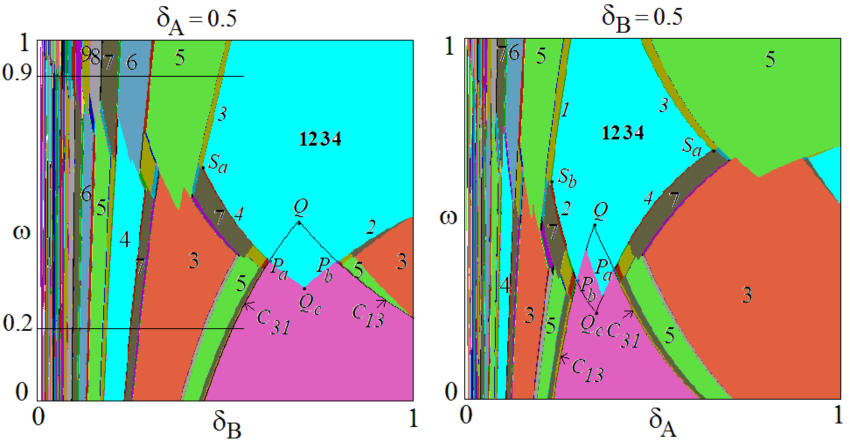
## Periodicity regions in a section of the parameter space

The period adding structure of the periodicity regions is clearly observable for $\delta_B < 1$. However, for $\delta_B = 1$ we have the period incrementing structure, only the principal regions exist, the regions are contiguous, the boundaries are those of the family symbolic sequence $\mathbf{134^n}$ for $n \geq 2$



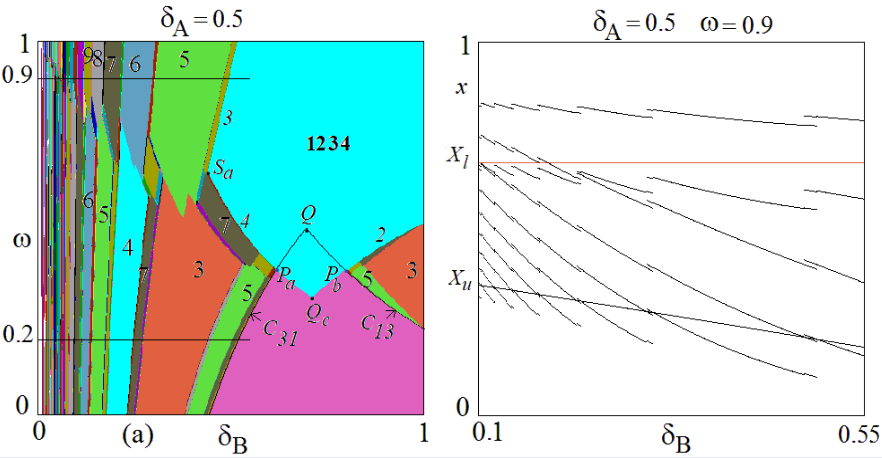
## Cycles of map F and related BCBs

The existence regions are interesting in the parameter plane $(\delta_B, \omega)$ or $(\delta_A, \omega)$, coexistence regions are better visible and all the boundaries can be seen



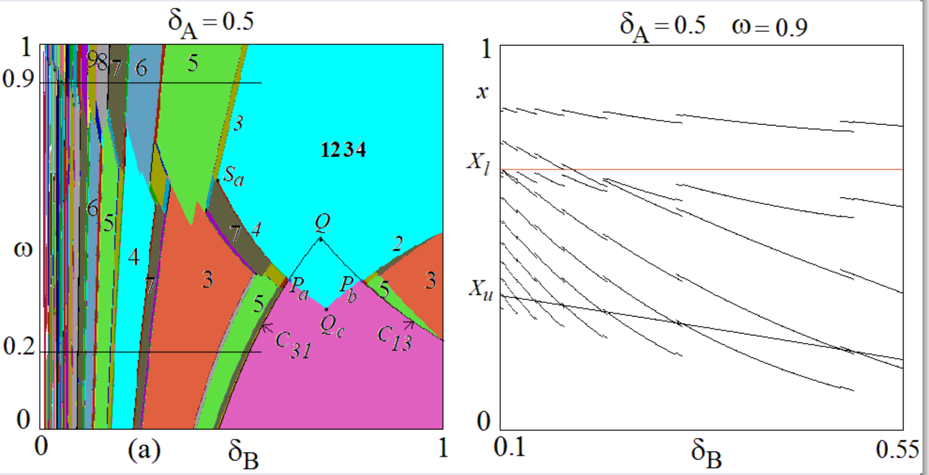
## Periodicity regions in a section of the parameter space

The period adding structure of the periodicity regions is clearly observable for $\delta_A < 1$ in the one-dimensional bifurcation diagram at $\omega = 0.2$



## Periodicity regions in a section of the parameter space

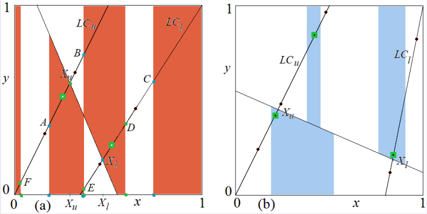
as well as in the one-dimensional bifurcation diagram at $\omega = 0.9$



# The 2D PWL system

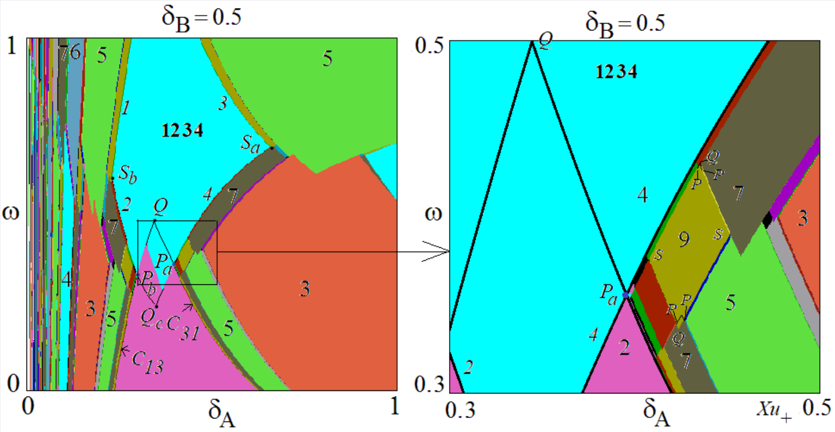
## Coexistence in the phase space

The basins are vertical strips up to the discontinuity line, the boundaries are given by $x-$values obtained by using the discontinuity points $X_u$ and $X_l$ and the related preimages on the two lines, we need only $F_u^{-1}(z) = \frac{z}{1-\delta_B}$, $F_l^{-1}(z) = \frac{z-\delta_A}{1-\delta_A}$




in (a) $\omega = 0.3$, $\delta_A = 0.35$ and $\delta_B = 0.5$, 2-cycle and 4-cycle,
$x(A) = F_l^{-1}(X_l) = \frac{X_l - \delta_A}{1-\delta_A}, \ldots, x(D) = F_u^{-1}(X_u) = \frac{X_u}{1-\delta_B}, \ldots$
In (b) $\omega = 0.7$, $\delta_A = 0.8$ and $\delta_B = 0.5$, 3-cycle and 5-cycle

## Coexistence in the parameter space



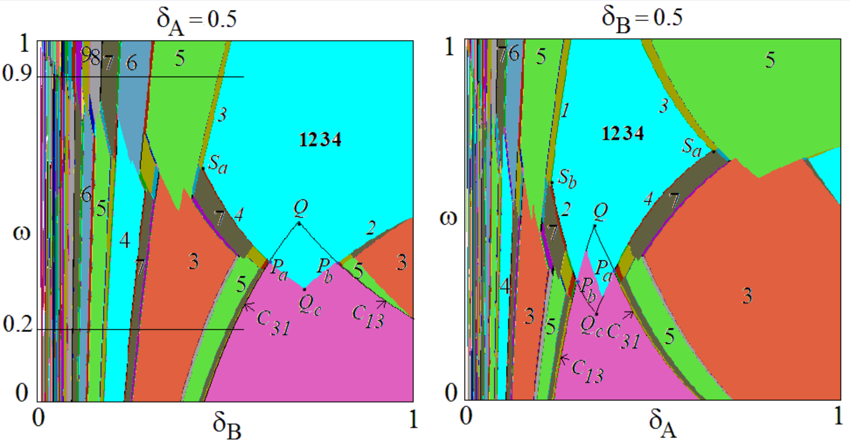
# The 2D PWL system

## Coexistence and quadrilateral regions in the parameter space

For the periodicity regions we have that the existence region of a cycle may be bounded by three or four BCB curves. The regions of overlapped parts are related to limit sets of periodicity regions having a quadrilateral shape (a cycle with $n \geq 4$ periodic points may have up to 4 border collision bifurcation boundaries, related to the discontinuity points $X_u$ and $X_l$ from below and from above, when all the 4 symbols are present in the symbolic sequence of the cycle),

- the borders of the periodicity regions related to an overlapped part, are not limit set of other periodicity regions,
- the borders of the periodicity regions not related to an overlapped part, are limit set of other periodicity regions,
- each periodicity region of quadrilateral shape has two opposite corners overlapped with another periodicity region, codimension-2 points of type-$Q$, and the other two corners are codimension-2 points of type-$S$. The four boundaries also include four codimension two points of type type-$P$

## Memory effect



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