Dynamics of a growth model with negative and positive externalities

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Negative and positive externalities

We consider a model with optimizing agents, in which economic dynamics is conditioned by:

- negative environmental externalities, which determine a reduction in net output via a damage function;
- positive externalities in the production process, which increase the productivity of physical capital and labour.

In such a context, the interplay between negative and positive externalities can give rise to several dynamical patterns, which we will describe extensively.

The model

Let K,P,L denote, respectively, the stock of physical capital, an index of environmental degradation and the representative agent's labour input. Assuming a Cobb-Douglas technology and the allocation of a share of the input to environmental defensive expenditures, the optimization problem the agent has to solve leads, eventually (see Borghesi et al., Urbino 2018), to the following dynamical system in the phase space $(K, P > 0, 0 < L < 1)$:

$$\dot{K} = \frac{1}{\vartheta(1+P^{\gamma})} K^{a+\alpha} L^{b+\beta-1}[(\vartheta+\beta)L - \beta]$$

$$\dot{P} = \delta K^{a+\alpha} L^{b+\beta} - \varepsilon P e^{-\zeta P}$$

$$\dot{L} = \frac{L(1-L)}{(1-b-\beta)(1-L) + \left(1 - \frac{\vartheta(1-\eta)}{\eta}\right)} \left\{(a+\alpha)\frac{\dot{K}}{K} - \gamma \frac{P^{\gamma-1}}{1+P^{\gamma}} \dot{P} + \frac{1}{\eta}\left(r - \frac{\alpha}{1+P^{\gamma}} K^{a+\alpha-1} L^{b+\beta}\right)\right\}$$

where K and P are state variables and L is the jumping variable. Moreover, all the parameters are positive, with α+β,b+β<1, 1≠η>(ϑ/(1+ϑ)).

Critical points and local analysis

The results about existence and local stability of critical points are summarized by the following

Theorem

- If a+α=1, there exists at most one critical point, whose Jacobian determinant is positive.
- If a+α>1, there exist at most two critical points, $E_1 = (K_1, P_1, L^*)$ and $E_2 = (K_2, P_2, L^*)$, with $K_1 > K_2, P_1 > P_2, L^* = \frac{\beta}{\vartheta+\beta}$, such that the Jacobian determinant is positive in E_1 and negative in E_2.
- If a+α<1, there exist at most two critical points, $E_1 = (K_1, P_1, L^*)$ and $E_2 = (K_2, P_2, L^*)$, with $K_1 > K_2, P_1 < P_2, L^* = \frac{\beta}{\vartheta+\beta}$, such that the Jacobian determinant is positive in E_1 and negative in E_2.

Global analysis

The first global analysis result concerns behaviour of the system *near* the boundaries of the phase space.

Theorem For every admittible set of parameters there exist both (positive) trajectories along which $(K,P,L) \to (0,0,0)$ (K,L doing so, possibly, in a finite time) and (positive) trajectories along which $(K,P,L) \to (\widehat{K},+\infty,1)$ (with $\widehat{K} = +\infty$ if $\gamma \leq 1$).

Three *open* regimes

It follows that, when an interior attractor exists, the system exhibits at least three *open* regimes (i.e., there exist three open regions where the trajectories have the same asymptotic behaviour).

This is the case, in particular, of two critical points, being, respectively, a saddle with two-dimensional stable manifold and a sink. In fact, we conjecture that in such a case there exist, generically, exactly three *open* regimes.

Then the question arises: what is the other surface, in addition to the stable manifold of the saddle, separating those regimes?

The answer requires a kind of "compactification" of the phase space: in words, we have to consider by "what slope", using an extension of language, certain trajectories tend to the boundary. Then the separating surfaces emerge (in the two cases $a + \alpha < 1$ and $a + \alpha > 1$) as "stable manifolds of boundary saddles".

The following theorem provides a detailed solution.

Separating surfaces

Theorem .

Let a +α < 1. Then the trajectories along which (P,L) tend to (+∞,1) (and K →+∞ as well if γ≤1) are separated from those along which K and P remain bounded as t > 0 by a surface which, after a change of variables, can be interpreted as the stable manifold of the "boundary saddle"

$$(K, K^{a+\alpha-1}P^{-\gamma}, L) = \left(0, \frac{r}{\alpha(L^*)^{b+\beta}}, L^*\right) \text{ if } \gamma\leq 1,$$

and of the "boundary saddle" $(KP^{\gamma-1}, K^{a+\alpha-1}P^{-\gamma}, L) = \left(0, \frac{r}{\alpha(L^*)^{b+\beta}}, L^*\right)$ if γ>1.

Moreover, among the latter trajectories, the stable manifold of E_1 separates those converging to (0,0,0) from those converging to E_2.

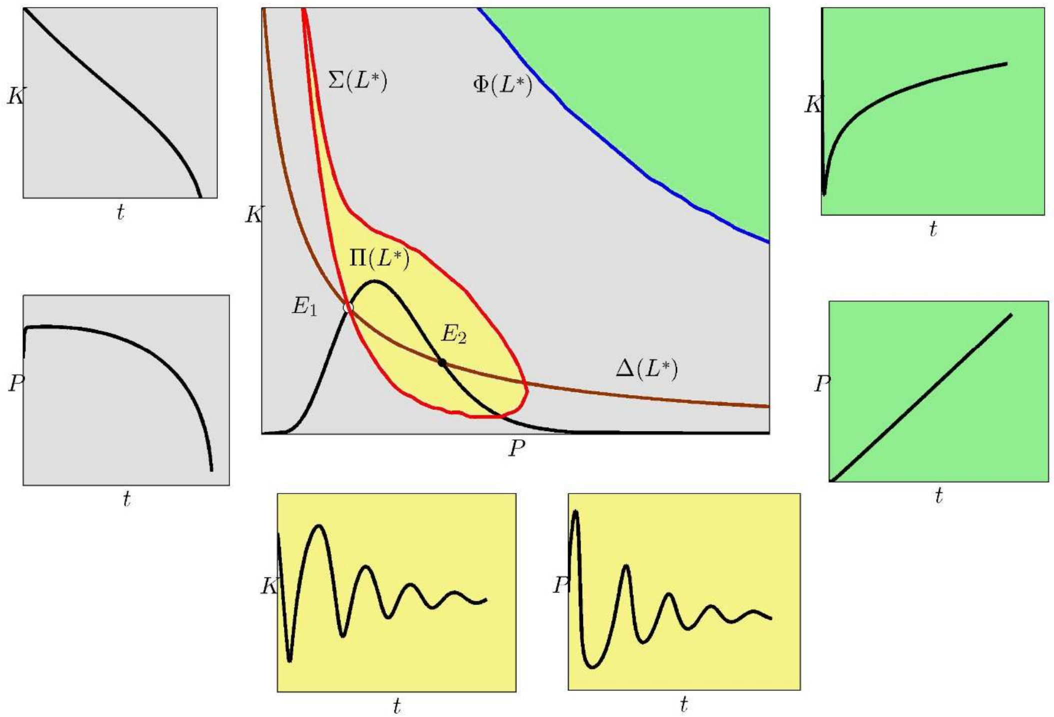
Let a+ α > 1. Then the trajectories converging to (K,P,L) = (0,0,0) are separated from those along which (K,P,L) are bounded away from zero as t > 0 by a surface which, after a change of variables, can be interpreted as the stable manifold of the "boundary saddle"

$$\left(K^{a+\alpha-1}L^{b+\beta-1}, L^{\frac{a+\alpha-b-\beta}{a+\alpha-1}}P^{-1}, L\right) = (m, n, 0)$$

where $m = r\frac{\vartheta}{\eta(\vartheta+\beta)}$, $n = \frac{m^{\frac{a+\alpha-1}{a+\alpha}}}{\delta}\max\left(\varepsilon - \frac{a+\alpha-b-\beta}{1-b-\beta}\frac{r}{\eta}, 0\right)$.

Moreover, among the latter trajectories, the stable manifold of E_1 separates those along which (P,L) →(+∞,1) (and K →+∞ if γ≤1) from those converging to E_2.

a+α<1. Basins in the plane L=L*



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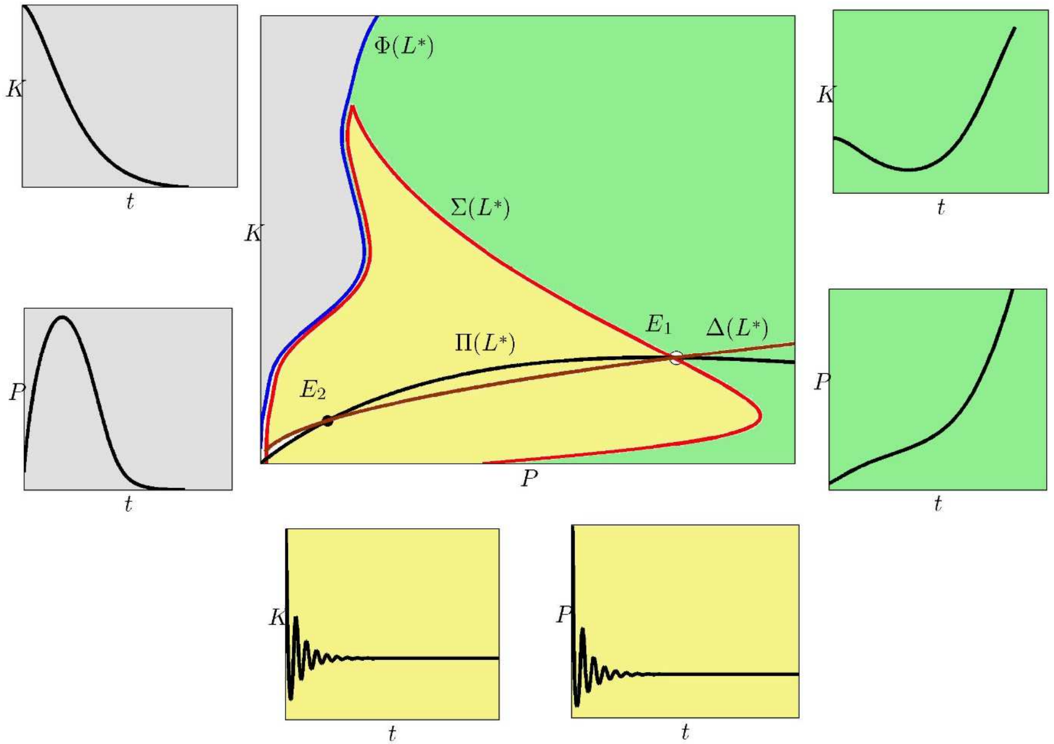


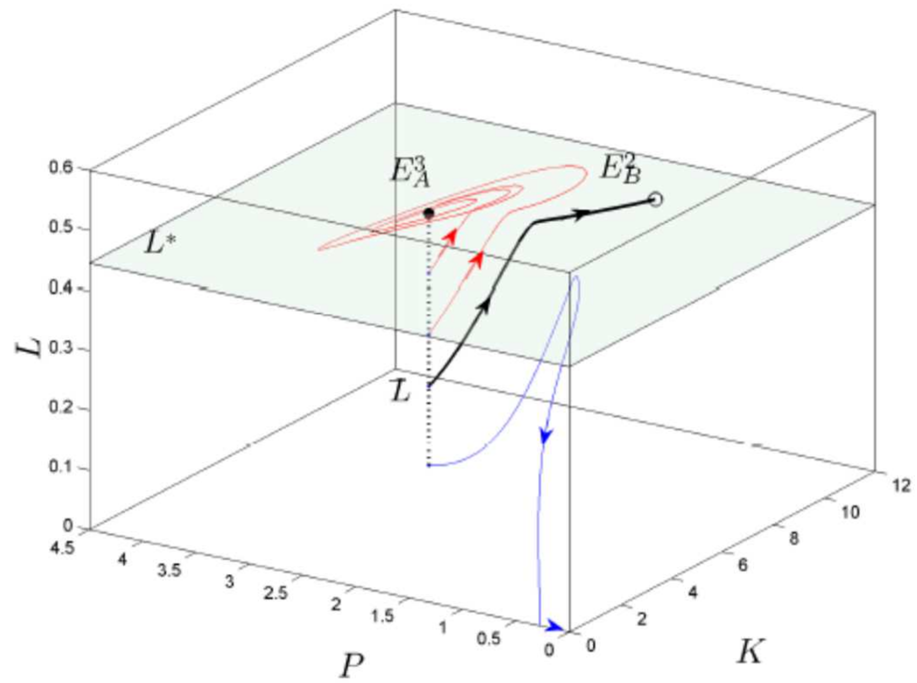
Indeterminacy

Assume, as above, one saddle, one sink and three *open* regimes exist. Then, in a neighbourhood of the saddle, to the same values of the state variables K and P and different values of the jumping variable L correspond points whose trajectories have different asymptotic outcomes. Precisely

Theorem Under the above assumptions, denote by Σ the stable manifold of E_1 and consider a sufficiently small neighbourhood U of E_1. Take $(K_0, P_0, \tilde{L}) \in U \cap \Sigma$. Then:

- If $a + \alpha < 1$, the trajectory starting from $(K_0, P_0, L_0) \in U$, with $L_0 > \tilde{L}$, converges to E_2, while the trajectory starting from $(K_0, P_0, L_0) \in U$, with $L_0 < \tilde{L}$, converges to (0, 0, 0).
- If $a + \alpha > 1$, the trajectory starting from $(K_0, P_0, L_0) \in U$, with $L_0 < \tilde{L}$, converges to E_2, while the trajectory starting from $(K_0, P_0, L_0) \in U$, with $L_0 > \tilde{L}$, converges to $(\hat{K}, +\infty, 1)$ ($\hat{K} = +\infty$ if $\gamma \leq 1$).

Indeterminacy: a+α<1



Indeterminacy: a+α>1