On the suppression mechanism of Bothwell's oscillations in Goodwin's business cycle

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Goodwin's economics cycle models

R.M. Goodwin, *The Nonlinear Accelerator and the Persistence of Business Cycles*, *Econometrica* **19**, 1-17 (1951)

The first Goodwin model is given by equations:

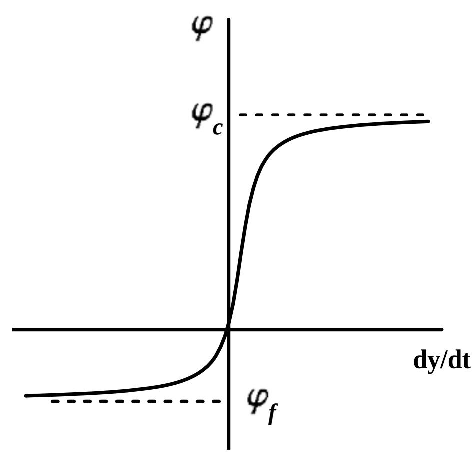
$$c(t) = \alpha y(t) + \beta(t), \quad \dot{k}(t) = l(t) + \varphi(\dot{y}(t)),$$

$$y(t) = \frac{1}{\varepsilon} \int_{-\infty}^{t} e^{-\frac{t-x}{\varepsilon}} \left(c(x) + \dot{k}(x) \right) dx.$$

Here $y(t)$ is income, $c(t)$ the consumption, $\beta(t)$ the autonomous components of consumption, $k(t)$ the capital stock, $\dot{k}(t)$ net investment, $\varepsilon > 0$ the time-lag of the dynamic multiplier, α the marginal propensity to consume, $0 \le \alpha \le 1$, $l(t)$ the autonomous components of investment, φ is the induced investment function, $\varphi'(\dot{y}) \ge 0$, $\varphi(0) = 0$, $\varphi'(0) = r > 0$,

$$\lim_{\dot{y} \to +\infty} \varphi(\dot{y}) = \varphi_c, \quad \lim_{\dot{y} \to -\infty} \varphi(\dot{y}) = \varphi_f,$$

r is the acceleration coefficient, φ_c and φ_f - the Hicksian ceiling and floor



These equations can be reduced to one

$$\varepsilon \dot{y}(t) + s y(t) = \varphi(\dot{y}(t)) + A(t), \qquad (1)$$

where $s = 1 - \alpha$, $A(t) = \beta(t) + l(t)$. Values of y, φ and A are expressed in billions of dollars per year. Time t is time in years, ε and r in years.

The initial condition must be consistent with the equation (1):

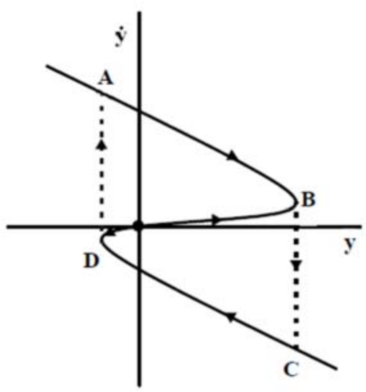
$$\varepsilon \dot{y}(0) + s y(0) = \varphi(\dot{y}(0)) + A(0)$$

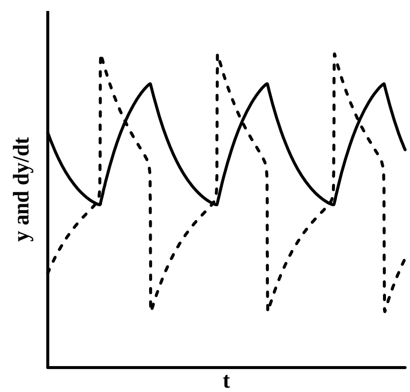
Goodwin's relaxation oscillations

If $A=A_0=\text{const}$,

$$y(t)=\frac{\varphi\left(\dot{y}(t)\right)-\varepsilon\dot{y}(t)+A_0}{s}.$$

For $r>\varepsilon$ relaxation oscillations are possible





Intuitive geometric arguments have been made rigorous by

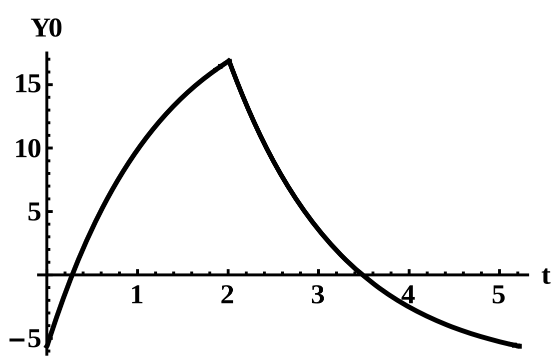
A. A. Andronov, A.A. Vitt and S.E. Khaikin. *Theory of oscillators* (1966)

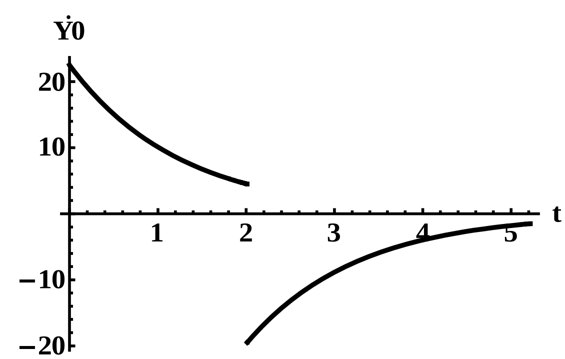
Analytical Solution for Piecewise Linear Delay Investment

$$\varphi_{PW}(\dot{y})=\begin{cases}\varphi_f\,,\ \dot{y}<r^{-1}\varphi_f\,,\\ r\dot{y},\ r^{-1}\varphi_c>\dot{y}\ge r^{-1}\varphi_f\,,\\ \varphi_c,\quad \dot{y}\ge r^{-1}\varphi_c.\end{cases}$$

Goodwins Parameters: $\varepsilon=0.5,\quad s=0.4,\quad r=2,\ \varphi_c=9,\ \varphi_f=-3$

$$Y0(t)=\begin{cases}22.5-28.125e^{-0.8t},\ 0\le t<2.01\\ -7.5+121.875e^{-0.8t},\ 2.01<t<5.22\end{cases}$$





Jumping behavior leads to discontinuous time dependence of *dy*/*dt* and to kinked time dependence of income *y* (*t*).

Goodwin's delay model with fixed time lag

Goodwin's hypothesis: If we take the nonlinear induced investment function with a fixed delay, $\varphi\left(\dot{y}(t-\theta)\right)$, then the modified equation (1),

$$\varepsilon\dot{y}(t)+sy(t)=\varphi\left(\dot{y}(t-\theta)\right)+A(t), \qquad (3)$$

will have a *continuous derivative* $\dot{y}(t)$. Here θ the time-lag between the investment decisions and the resulting outlays.

Goodwin's 2nd order ODE model

Formally, equation (3) can be represented as

$$\varepsilon\dot{y}(t+\theta)+sy(t+\theta)=\varphi\left(\dot{y}(t)\right)+A(t+\theta), \qquad (4)$$

Goodwin approximated (4) upon replacing $y(t+\theta)$ and $\dot{y}(t+\theta)$ by the first two terms of their Taylor's expansion

$$y(t+\theta)\approx y(t)+\theta\dot{y}(t), \quad \dot{y}(t+\theta)\approx \dot{y}(t)+\theta\ddot{y}(t)$$

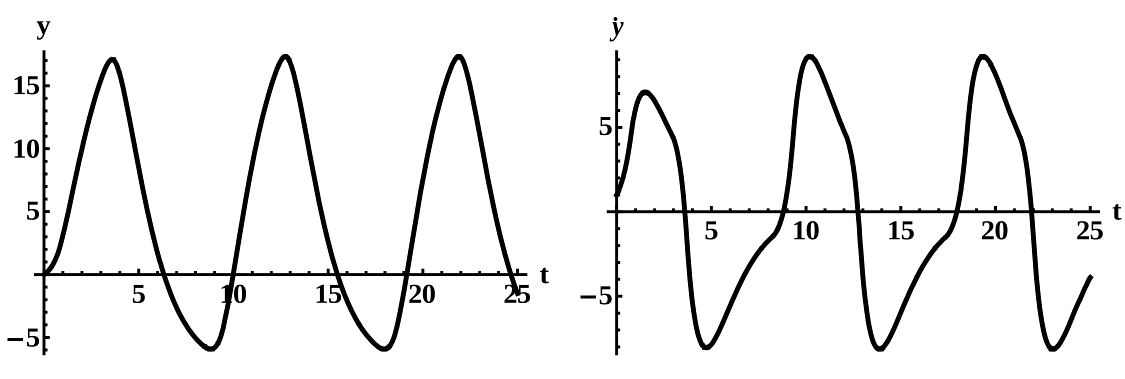
and obtained the nonlinear ODE of the Lord Rayleigh type

$$\varepsilon\theta\ddot{y}(t)+\left(\varepsilon+s\theta\right)\dot{y}(t)+sy(t)=\varphi\left(\dot{y}(t)\right)+A(t)+\theta\frac{dA(t)}{dt} \qquad (5)$$

Goodwin suggested that $A = A_0$ = const and showed that if the inequality

$$r > \varepsilon + s\theta$$

holds, then equation (5) admits a solution in the form of a stable limit cycle.



	Period	y_{max}	y_{min}	$\dot{y}_{max}$	$\dot{y}_{min}$	$\langle y \rangle$
θ=0	5.22	16.88	-5.63	22.5	-19.5	4.067
θ=1	9.16	17.33	-5.93	9.2	-8.13	4.31

S. Sordi, *"'Floors' and/or 'Ceilings' and the Persistence of Business Cycles", 2006*

Goodwin's delay model with fixed time lag. Multiplicity of solutions

$$\varepsilon \dot{y}(t) + sy(t) = \varphi\left(\dot{y}(t-\theta)\right) + A_0$$

Initial conditions

We must specify the initial function y=Φ(t) that determines behavior of y for - θ≤ t ≤ 0.

This equation is DDE of a neutral type and, according to the theory

L. E. Els'golts and S. B. Norkin, *Introduction to the Theory and Application of Differential Equations with Deviating Arguments, 1973*

the solution strongly depends on the initial function. Even if $y(0) = \Phi(0)$ we have that, the right-hand derivative $\dot{y}(0)$, is different in general from the left-hand derivative $\dot{\Phi}(0)$. This irregularity at t=0 usually propagates along the integration interval. In general, this creates a further jump discontinuity in the first derivative of the solution $y(t)$ at t=θ, and so on. And only if the splicing conditions are satisfied,

$$\varepsilon\dot{\Phi}(0) + s\Phi(0) = \varphi\left(\dot{\Phi}(-\theta)\right) + A(0)$$

the derivative will be continuous.

Linearized equation

Let A =A_0. Linearizing the equation about stationary value $y_s = s^{-1}A_0$,

$$y = y_s + \delta y,\ \delta y \ll y_s,\ \ \delta y \sim e^{\lambda t}$$

we obtain the linear neutral DDE

$$\varepsilon\delta\dot{y}(t) + s\delta y(t) = r\delta\dot{y}(t-\theta)\ .$$

Putting $\delta y \sim e^{\lambda t}$, we find

$$\lambda + \frac{s}{\varepsilon} = \lambda e^{-\lambda\theta} \frac{r}{\varepsilon}$$

Els'golts , Norkin:

For $|\lambda| \gg 1$

$$\lambda \approx \lambda e^{-\lambda\theta} \frac{r}{\varepsilon} \rightarrow \; e^{\lambda\theta} = \frac{r}{\varepsilon}$$

$$\lambda_n = \frac{1}{\theta} \ln \frac{r}{\varepsilon} + \frac{2\pi m}{\theta} i, \; m = \pm 1, \ldots$$

If $r > \varepsilon$, there are an infinite number of unstable solutions

$$\delta y = e^{\gamma t} \left(C_1 \cos \frac{2\pi m}{\theta} t + C_2 \sin \frac{2\pi m}{\theta} t \right)$$

with the same increments $\gamma = \frac{1}{\theta} \ln \frac{r}{\varepsilon}$.

F. Bothwell *The Method of Equivalent Linearization* (1952):

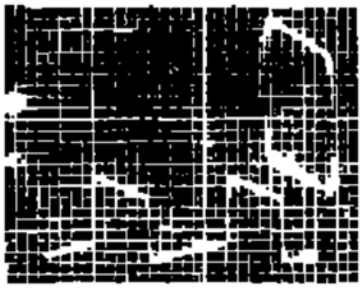
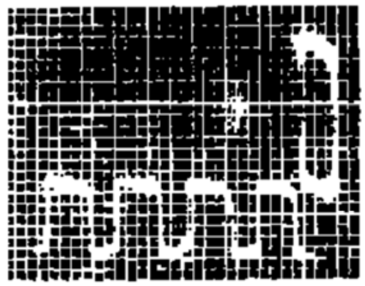
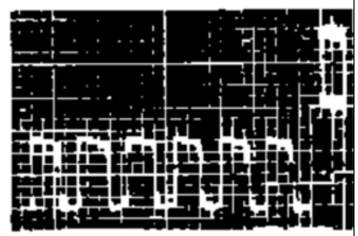
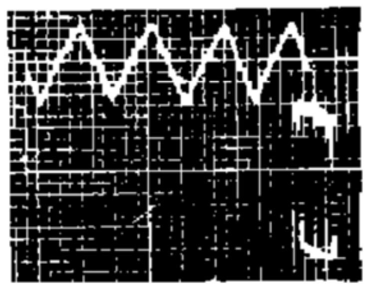
- For $r > \varepsilon$ there are an infinite number of nonlinear steady state solutions of Goodwin's equation with fixed delay, $y_{Bn}(t)$
- $y_{Bm}(t) = a_{Bm} - b_{Bm} \cos \frac{2\pi t}{T_{Bm}}, m = 0, 1, .2, ..$

  m=0 - primary mode (Goodwin's mode)
  m ≥1 - infinite number of secondary modes
- For **Goodwins Parameters** and θ=1

$$T_{B0} \approx 7.93, \; b_{B0} \approx 13.4, \; a_{B0} \approx 6.01$$

$$T_{Bm} \approx \frac{1}{m}, \; b_{Bm} \approx \frac{2.4}{m}, \; a_{Bm} \approx 6.48, \; m = 1, .2, ..$$

By analogue computer modelling,
R. H. Strotz, J. C. McAnulty and J. B. Naines *Goodwin's Nonlinear Theory of the Business Cycle: An Electro-Analog Solution* (1953), confirmed Bothwell's conclusions about the existence the short-periodic oscillations. But the shape of the oscillations was fundamentally different from the cosine: these were oscillations of relaxation type with discontinuous time derivative $dy(t)/dt$.

Mode 1	Mode 2	Mode3
 Photo 7	 Photo 10	 Photo 13
		

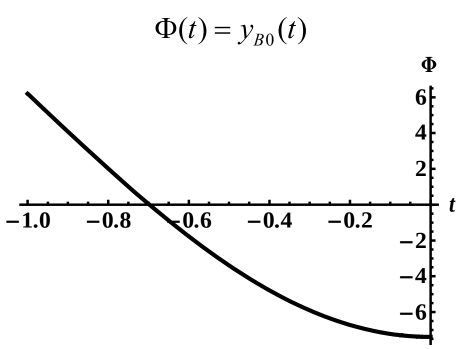
Bothwell and Strotz et al. do not inform about the form of initial function, leading to the establishment of mode 1, 2, ...

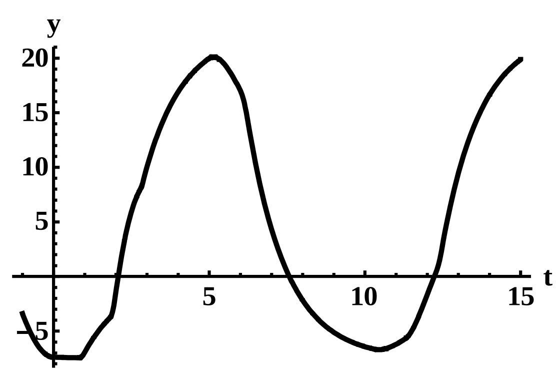
We assumed that Strotz et al. chose an initial function close to Boswell's mode itself

$$\Phi_m(t) = y_{Bm}(t), \;\; m = 1, \, 2, ...$$

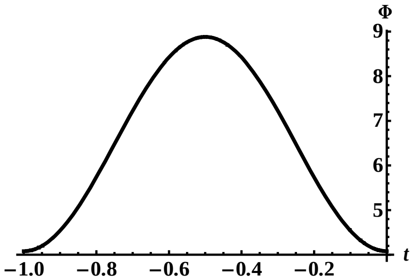
(ideal initial conditions for excitation of mode m). The direct numerical modeling confirmed this assumption.

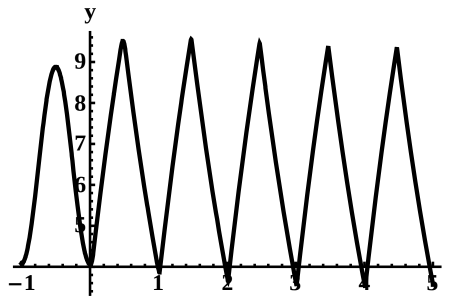
m=0



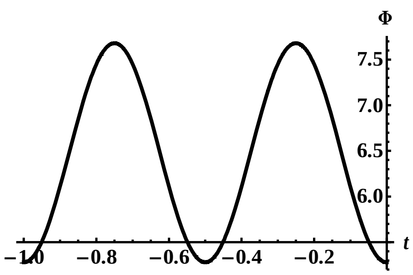


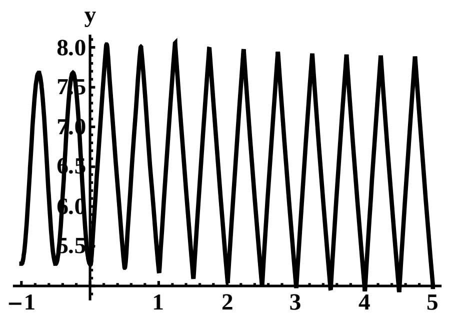
m=1 $\Phi(t)=y_{B1}(t)$





m=2 $\Phi(t)=y_{B2}(t)$



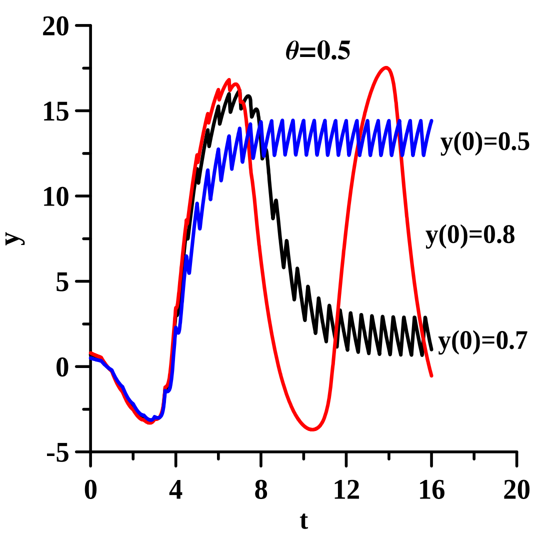


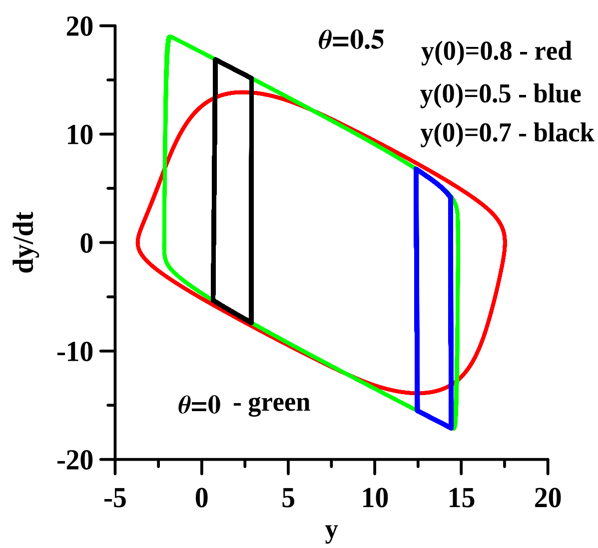
In this way, we can obtain numerically the short-period modes with any *m*.

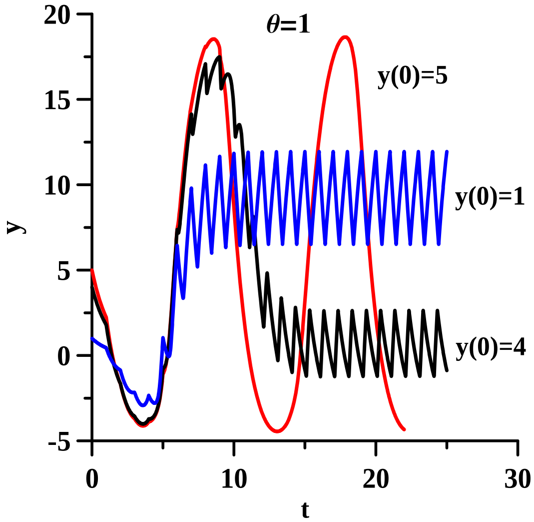
Excitation DDE oscillations by monotonic initial function

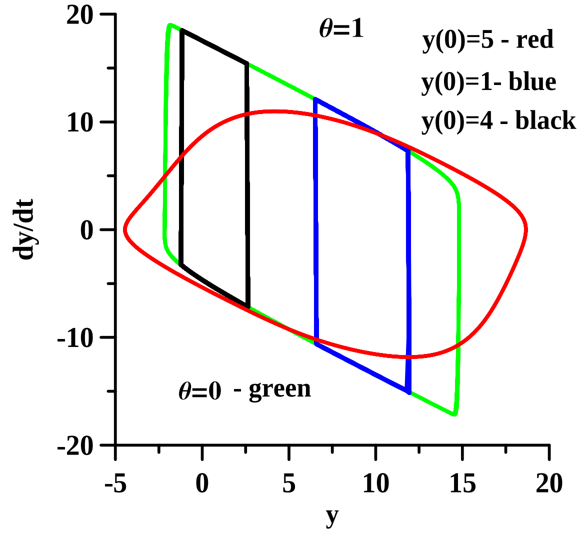
However, it is easy to emphasize that monotonic initial functions can also lead to the establishment of not only the Goodwin mode *m*=0, but also the first mode, *m*=1. Competition between modes *m*=0 and *m*=1 in the transition process can greatly affect the steady state shape of first mode.

$$\varepsilon=0.5,\quad s=0.4,\quad \theta=1,\quad \varphi(\dot y)=\frac{12}{\pi}\left(\arctan(\dot y-1)+\frac{\pi}{4}\right),\quad \Phi(t)=y(0)=const$$

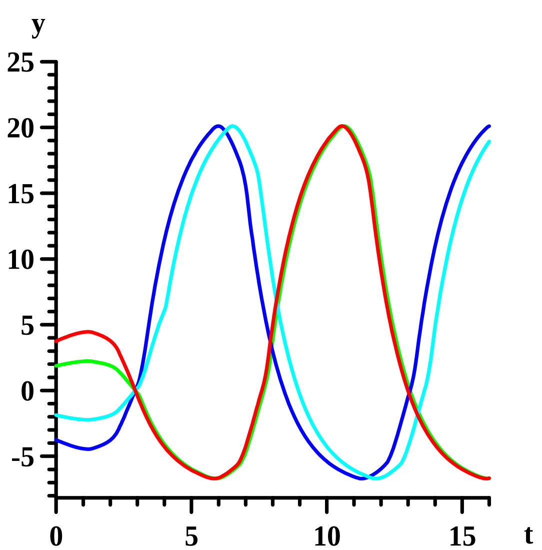




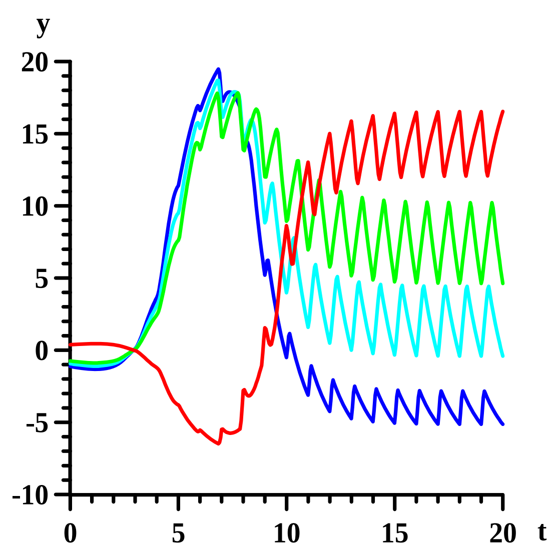




$$\varepsilon = 0.5, \quad s = 0.4, \quad \theta = 1, \quad \varphi_S(\dot{y}) = \frac{9 - 9\exp\left(-\frac{8\dot{y}}{9}\right)}{1 + 3\exp\left(-\frac{8\dot{y}}{9}\right)}, \quad \Phi(t) = \mathrm{v}_0\left(t + 3.75\right)$$



Blue lines correspond to the value $v_0 = -1$, cyan – to $v_0 = -0.5$, green – to $v_0 = 0.5$ and red – to $v_0 = 1$



Blue lines correspond to the value $v_0 = -0.3$, cyan – to $v_0 = -0.25$, green – to $v_0 = -0.2$ and red – to $v_0 = 0.1$

Analytical solutions of DDE with $\varphi(\dot{y}) = \varphi_{PW}(\dot{y})$

Bothwell		Antonova et al 2013	Matsumoto et al 2018
	$\Phi(t) = y_{B1}(t)$	$\Phi(t) = t$	$\Phi(t) = -2$
$y_{\max} = 8.08,$ $y_{\min} = 4.08,$ $\langle y \rangle = 6.46$	$y_{\max} = 9.36,$ $y_{\min} = 3.54,$ $\langle y \rangle = 6.36$	$y_{\max} = 13.2,$ $y_{\min} = 7.56,$ $\langle y \rangle = 10.45$	$y_{\max} = 12.44,$ $y_{\min} = 6.57,$ $\langle y \rangle = 9.56$

- For all initial functions that lead to excitation of the primary mode, the steady-state solutions differ only in phase shift

- For all the initial functions that lead to excitation of mode 1, the steady-state solutions differ in average value $\langle y(t) \rangle$, maximal and minimum values $y_{\max}, y_{\min}$

Models with continuously distributed lag

Goodwin model with continuously distributed lag can be written in the form (**R. G. D. Allen**, *Mathematical economics*, (1964).)

$$\begin{cases} \varepsilon\, \dot{y}(t) + sy(t) = I(t) + A(t),\ t > 0, \\ I(t) = \int\limits_{-\infty}^{t} f(t-x)\varphi\left(\dot{y}(x)\right)dx, \quad y(t) = \Phi(t),\ -\infty \le t \le 0. \end{cases} \qquad (6)$$

where $f(x) \geq 0$ is the delay kernel, $\int\limits_{0}^{\infty} f(x)dx = 1$.

If $f(x)$ is taken to be a Dirac distribution, i.e. $f(x) = \delta(x-\theta)$, we obtain the model with fixed delay (Eq.(3)). If $f(x) = f_e(x) = \theta^{-1} e^{-\frac{x}{\theta}}$, we obtain the ODE Goodwin's model, since

$$I(t) = \frac{1}{\theta} \int\limits_{-\infty}^{t} e^{-\frac{t-x}{\theta}} \varphi\left(\dot{y}(x)\right)dx$$

satisfies the differential equation

$$\theta \dot{I}(t) + I(t) = \varphi\left(\dot{y}(t)\right),\ I(0) = \frac{1}{\theta} \int\limits_{-\infty}^{0} e^{x/\theta} \varphi\left(\dot{\Phi}(x)\right)dx\,.$$

We get a system of equations

$$\begin{cases} \varepsilon\, \dot{y}(t) + sy(t) = I(t) + A(t), \\ \theta \dot{I}(t) + I(t) = \varphi\left(\dot{y}(t)\right) \end{cases}$$

that reduces exactly to the 2nd order Goodwin ODE and secondary oscillations would not arise.

The most important characteristics of $f(x)$ are the average delay time T_d, its variance σ^2 and coefficient of variation V

$$T_d = \int_0^\infty s f(s) ds, \; \sigma^2 = \int_0^\infty (s - T_d)^2 f(s) ds, \quad V = \frac{\sqrt{\sigma^2}}{T_d}.$$

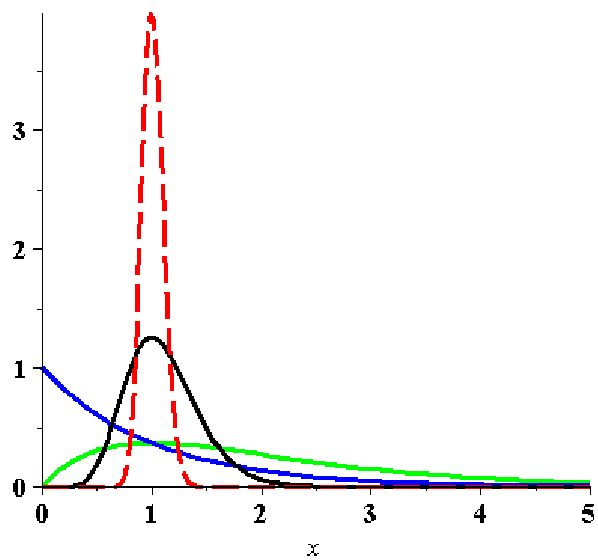
If $f(x) = f_e(x)$, $T_d = \sigma = \theta$, $V = 1$. If $f(x) = \delta(x - \theta)$, $T_d = \theta$, $\sigma = 0$, $V = 0$

We see, that the number of excited modes depends on the half-width of the delay kernel (or coefficient of variation).

In this work we will analyze this dependence. We will consider two delay kernel: with Gamma distribution and with uniform distribution

Model with Gamma distribution delay kernel

$$g_k(s,\theta) = \frac{1}{\theta (k-1)!} \left(\frac{ks}{\theta} \right)^k \exp\left(-\frac{ks}{\theta} \right), \quad k = 1,2,...$$



blue - $f(x) = f_e(x)$, green - m=1, black - m=10, red - m=100

$$T_d = \frac{(m+1)}{m}\theta, \quad \sigma^2 = \frac{(m+1)}{m^2}\theta^2, \quad V = \frac{1}{\sqrt{m+1}}$$

Properties

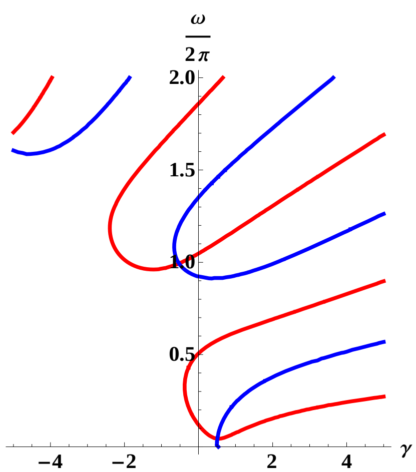
$$\lim_{m \to \infty} \int_{-\infty}^{t} g_m(t-s,\theta) X(s) ds = X(t-\theta), \qquad \left(\frac{d}{dt} + \frac{m}{\theta} \right)^{m+1} g_m(t,\theta) = 0$$

The integro-differential equation (6) is equivalent to the k+2 order ODE system

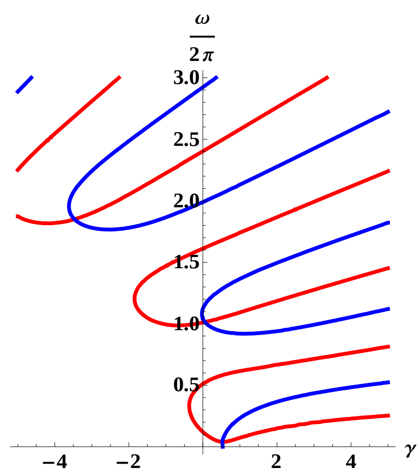
$$\begin{cases} \varepsilon\, Dy(t) + (1-\alpha)\, y(t) = I(t) + A_0, \quad D = \dfrac{d}{dt} \\ \left(1 + \dfrac{\theta D}{k}\right)^{k+1} I(t) = \varphi\big(D(t)\big), \quad k = 1, 2, \ldots \end{cases}$$

Characteristic equation

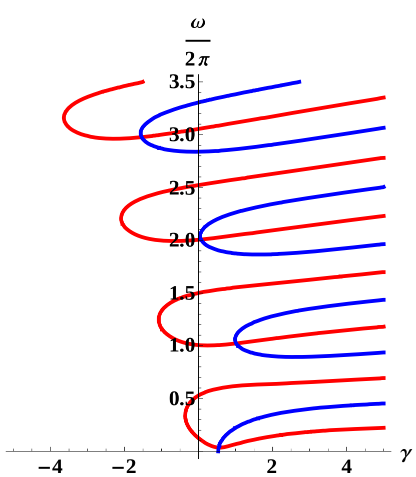
$$\Delta_g(\lambda) = (\varepsilon\lambda + s)\left(1 + \frac{\theta\lambda}{k}\right)^{k+1} - r\lambda = 0$$



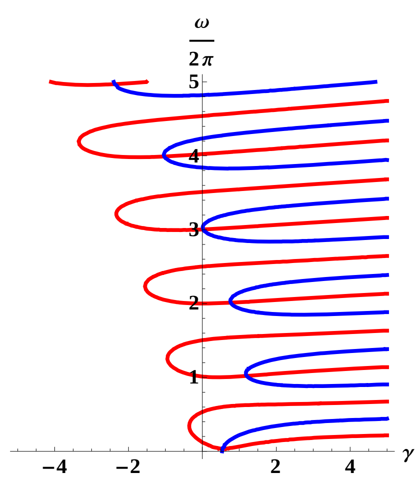
k=10



k=15



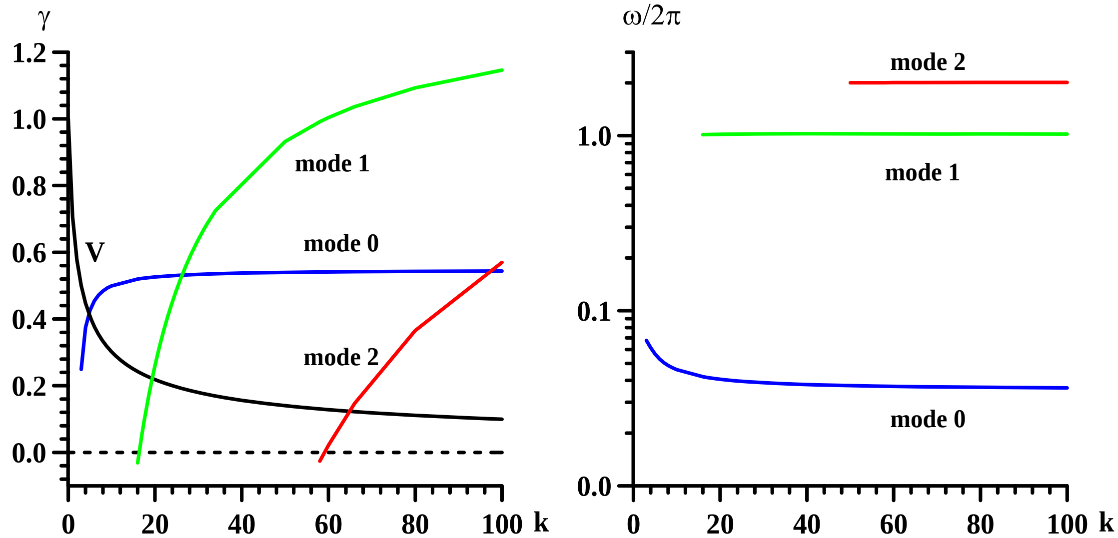
k=60



k=130

$\lambda = \gamma + i\omega$, coutours $\operatorname{Re}\Delta_g(\gamma,\omega) = 0$ (red) and $\operatorname{Im}\Delta_g(\gamma,\omega) = 0$ (blue),

$\varepsilon = 0.5$, $s = 0.4$, $\theta = 1$, $r = 2$



Model with uniform distribution delay kernel

Our assumption about the form of $f(x)$ is that the delay times are symmetrically distributed around a mean value θ between the minimum and maximum delays,

$$\theta_{min} = \theta - \tau\,,\ \ \theta_{max} = \theta + \tau, \quad \tau \ll \theta,$$

$$f_u(x) = \frac{1}{2\tau}\left[H(x-\theta_{\min}) - H(x-\theta_{\max})\right],$$

$$T = \theta,\ \sigma = \frac{\tau}{\sqrt{3}} \text{ and } V = \frac{\tau}{\sqrt{3}\theta} \ll 1,$$

where $H(x)$ is the Heaviside function, τ denotes the half width of delay-kernel. Then from the equations (6), we get

$$\varepsilon\, \dot{y}(t) + s y(t) = \frac{1}{2\tau}\int_{t-\theta-\tau}^{t-\theta+\tau} \varphi\left(\dot{y}(x)\right) dx.$$

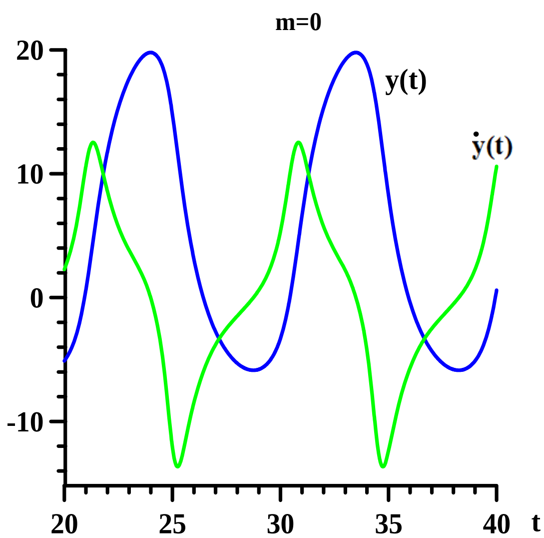
If we differentiate both sides with respect to t, we get

$$\varepsilon\, \ddot{y}(t) + s\dot{y}(t) = \frac{\varphi\left(\dot{y}(t-\theta+\tau)\right) - \varphi\left(\dot{y}(t-\theta-\tau)\right)}{2\tau} + \frac{dA(t)}{dt} \qquad (7)$$

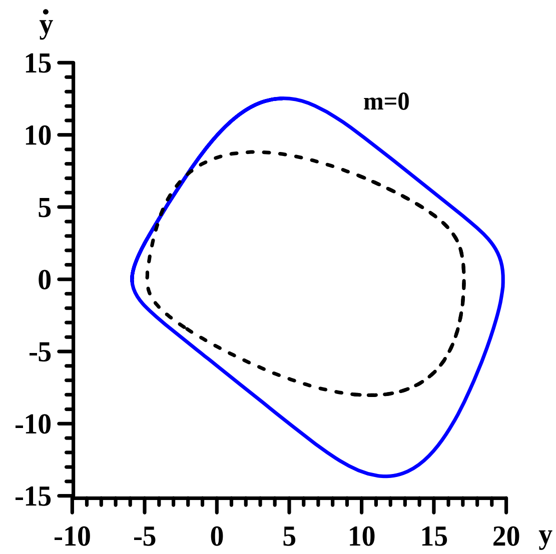
Eq. (7) is a retarded DDE with two fixed delays $\theta - \tau$ and $\theta + \tau$.

Numerical results: $\varepsilon = 0.5, \quad s = 0.4, \quad \theta = 1, \quad \varphi_s(\dot{y}) = \dfrac{9 - 9\exp\left(-\dfrac{8\dot{y}}{9}\right)}{1 + 3\exp\left(-\dfrac{8\dot{y}}{9}\right)},$

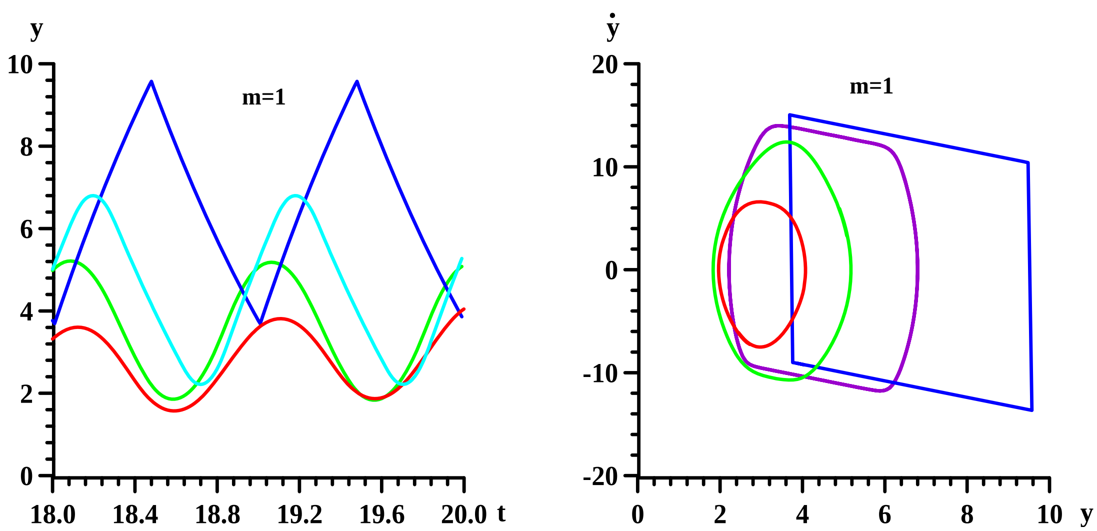
$$\Phi_m(t) = y_{Bm}(t), \; m = 0, 1, 2, 3,$$



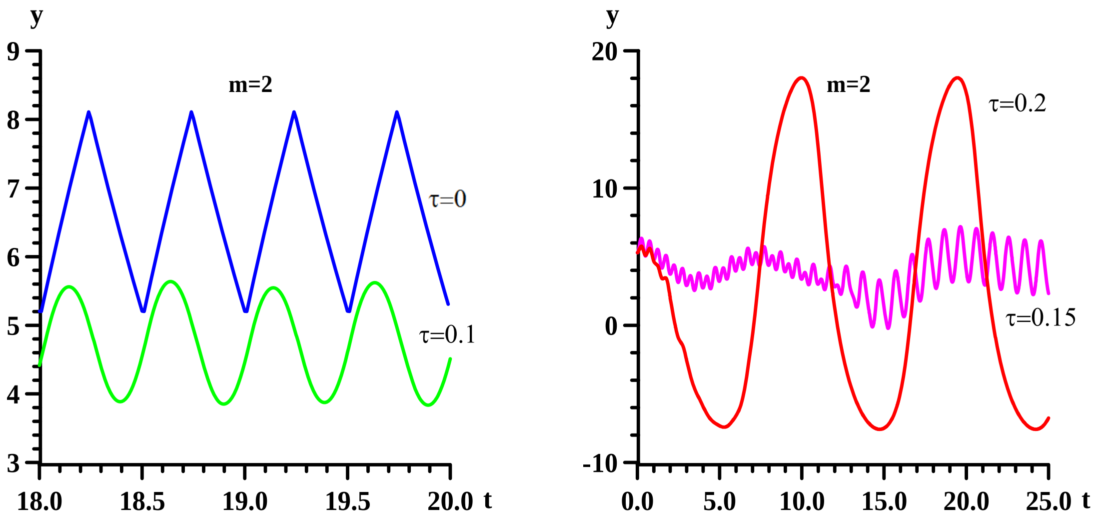
m=0, τ=0…0.3: blue line – y(t), green – dy/dt.



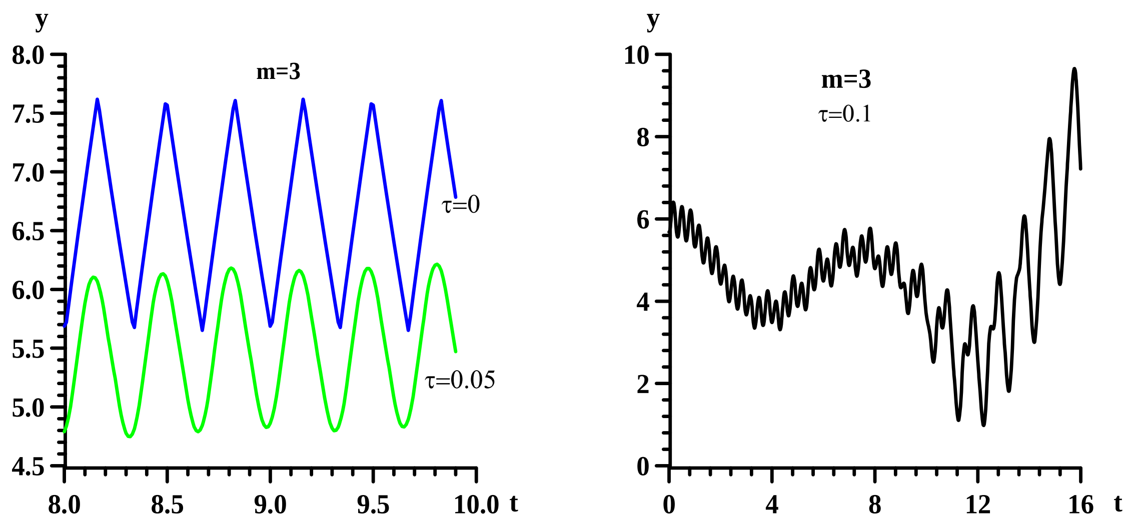
m=0, τ=0…0.3: blue line – phase curve $\dot{y}(y)$, dotted line - $\dot{y}(y)$ for 2[nd] order ODE model



Mode m=1, τ=0 (blue), 0.1 (cyan), 0.2(green), 0.3(red)



Mode 2



Mode 3

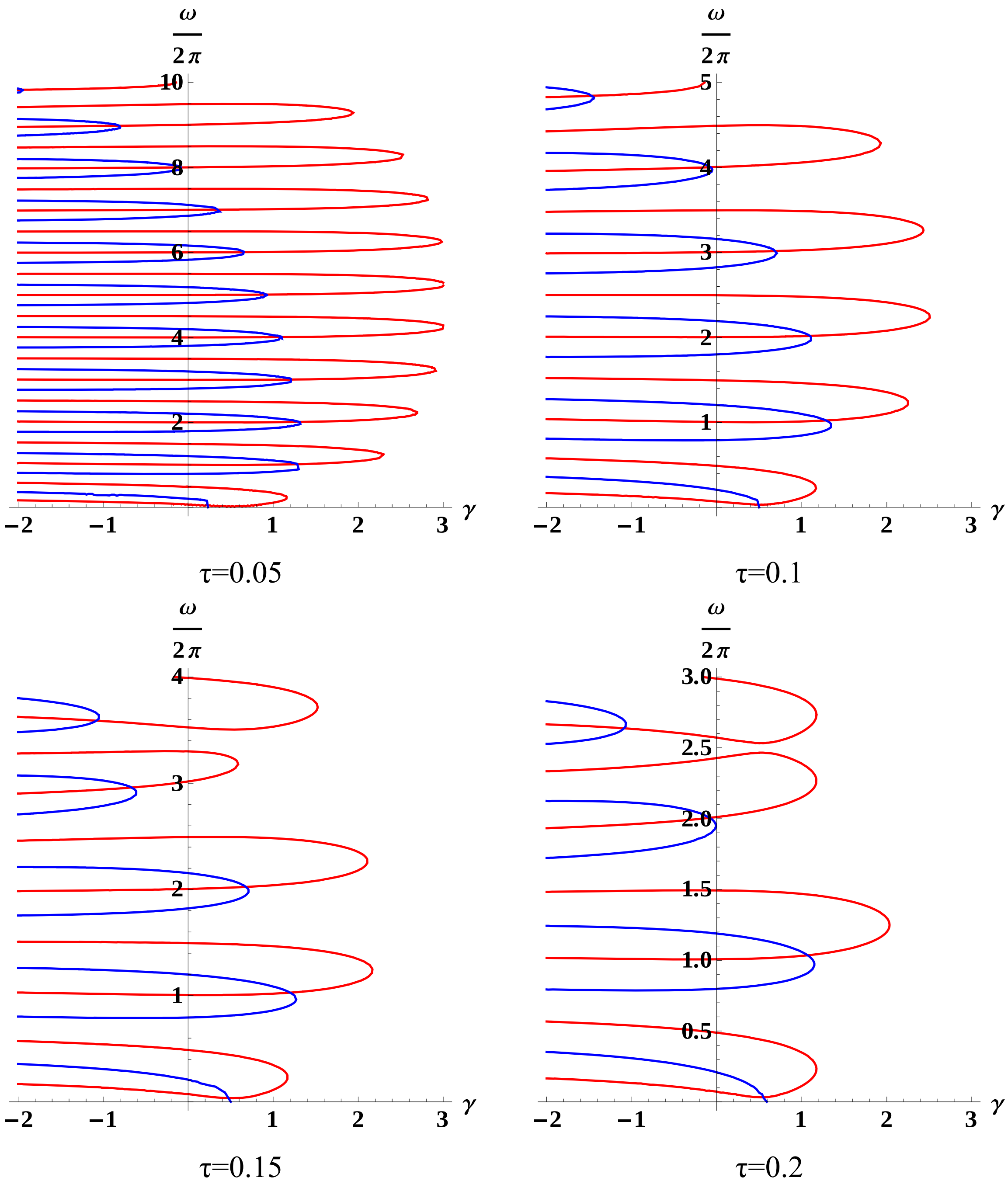
Behavior of solutions near a stationary point

Linearizing (7) about $\dot{y}=0$ we obtain

$$\varepsilon\delta\ddot{y}\left(t\right)+s\delta\dot{y}\left(t\right)=r\frac{\delta\dot{y}(t-\theta+\tau)-\delta\dot{y}(t-\theta-\tau)}{2\tau}$$

and the corresponding characteristic equation

$$\Delta_u(\lambda)=\lambda\varepsilon+s-re^{-\lambda\theta}\frac{\sinh\lambda\tau}{\tau}=0$$



$\lambda = \gamma + i\omega$, coutours $\mathrm{Re}\Delta_u(\gamma,\omega) = 0$ (red) and $\mathrm{Im}\Delta_u(\gamma,\omega) = 0$ (blue)

Some roots of the characteristic equation $\Delta_u(\lambda)=0$ for $\varepsilon = 0.5$, $s = 0.4$, $r = 2$, $\theta = 1$

m	τ=0	τ=0.05	τ=0.1	τ=0.15
0	0.55±0.22i	0.55±0.22i	0.55±0.22i	0.55±0.22i
1	1.36±6.34i	1.34±6.41i	1.29±6.43i	1.20±6.46i
2	1.38±12.63i	1.31±12.64i	1.1±12.68i	0.68±12.72i
3	1.38±18.89i	1.23±18.91i	0.69+18.95i	-0.66±18.59i
4	1.38±25.16i	1.11±25.19i	-0.07±25.15i	

Suppose that $\lambda = i\omega,\ \omega > 0$. From

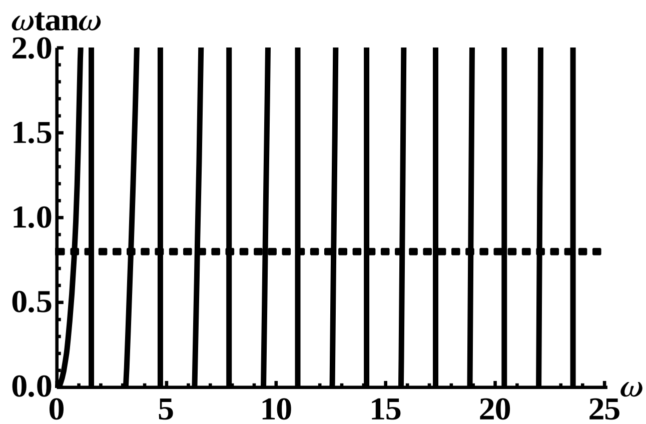
$$\operatorname{Re}\Delta_u = s - \frac{r\sin\omega\tau}{\tau}\sin\omega\theta = 0,$$

$$\operatorname{Im}\Delta_u = \varepsilon - \frac{r\sin\omega\tau}{\omega\tau}\cos\omega\theta = 0,$$

we get the system of equations on ω and τ.

Equation for ω:

$$\omega\tan\omega\theta = \frac{s}{\varepsilon},\ \ \cos\omega\theta > 0\ !$$



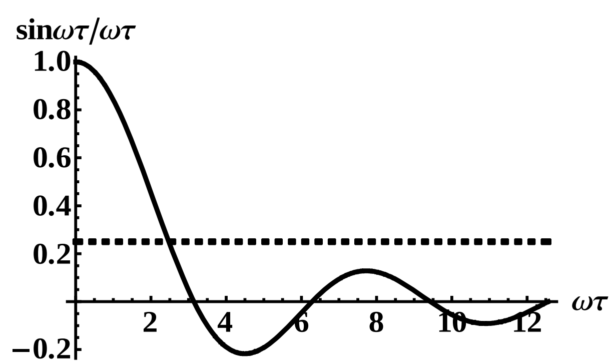
Solutions for $m \geq 1$

$$\omega_m \approx \frac{2\pi m}{\theta}.$$

Equation for τ

$$\frac{\sin\omega_m\tau}{\omega_m\tau} = \frac{1}{r}\sqrt{\varepsilon^2 + \frac{s^2}{\omega_m^2}} \approx \frac{\varepsilon}{r},$$

since $0<s<1$, $\frac{s}{\omega_m} \ll 1$ and $\varepsilon \gg \frac{s}{\omega_m}$.



$$\frac{\sin\omega_m\tau}{\omega_m\tau}=\frac{\varepsilon}{r}=0.25, \qquad \tau_m\approx\frac{2.475}{\omega_m}=\frac{0.39}{m}\theta.$$

If $\tau\neq 0$, then a finite number of modes are excited (approximately)

$$m_*\approx\left[0.39\frac{\theta}{\tau}\right].$$

All mods with $m > m_*$ will be stable.

Frequencies ω_m and threshold values of τ_m for $\varepsilon=0.5,\ s=0.4,\ r=2,\ \theta=1$

m	1	2	3	4
ω_m	6.41	12.63	18.892	25.164
T_m	0.98	0.497	0.333	0.25
τ_m	0.39	0.196	0.131	0.098
$\tau_m\frac{d\gamma}{d\tau}(\gamma=0,\omega_m,\tau_m)$	-3.21	-3.85	-4	-4.07

Excitation DDE oscillations by monotonic initial function

If initial function is monotonic, then the modes with $m\geq 2$, are not excited. Let $\Phi=at$, $a\neq 0$. Then mode m=1 is excited, if $a_{\min}<a<a_{\max}$; if $a_{\min}>a$ and $a>a_{\max}$, then mode m=0 is excited

τ	0	0.05	0.1
$a_{\min}$	-0.65	-0.12	-0.07
$a_{\max}$	1.79	0.234	0.153

Conclusions

- Goodwin's model with uniform distribution delay kernel investigated

- We have answered some questions that remained unexplained in Boswell's and Strotz's et al. papers

Bothwell ([1], p. 282): "*Theoretically, there can exist periods of the order of magnitude of a year, a month, a day, a second, or even a microsecond. Of course, the very short periods violate the conditions under which the model was formulated and may be excluded on that basis. But where should the line of exclusion be drawn? Is the one-year period legitimate, and, if so, how about the six-month period*?"

Strotz, et.al., (1953, p: 408)
It was difficult to obtain any given mode higher than the fourth and, whenever once obtained, it would non persist for long. On the other hand, the first mode easily obtained and would persist for hours.

Assuming that τ/θ is about 0.1 for ideal initial functions we can observe the modes with periods 1, $\frac{1}{2}$ and $\frac{1}{3}$. Modes with periods $\frac{1}{4}$, $\frac{1}{5}$… are damped.

- For monotonic initial function even mode with period 1 is hard to implement.

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Thank you very much for your attention!